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CORRELATED EQUILIBRIA IN VOTER TURNOUT GAMES

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Abstract

Communication is fundamental to elections. This paper extends canonical voter turnout models to include any form of communication, and characterizes the resulting set of correlated equilibria. In contrast to previous research, high-turnout equilibria exist in large electorates and uncertain environments. This difference arises because communication can be used to coordinate behavior in such a way that voters find it incentive compatible to always follow their signals past the communication stage. The equilibria have expected turnout of at least twice the size of the minority for a wide range of positive voting costs, and show intuitive comparative statics on turnout: it varies with the relative sizes of different groups, and decreases with the cost of voting. This research provides a general micro foundation for group-based theories of voter mobilization, or voting driven by communication on a network.

1 Introduction

What drives voter turnout is a fundamental question in political economy. Canonical models, which rely on voters rationally and independently deciding whether to turn out based on how likely they are to be pivotal to the election outcomes, provide unsatisfactory

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explanations (Downs (1957), Riker and Ordeshook (1968), Palfrey and Rosenthal (1985), Myerson (2000)). In particular, these models fail to rationalize the high turnout rates observed in very large elections. Intuitively, as the electorate grows large, the probability that any individual voter is pivotal goes to zero, so with voting incurring a cost, very few people should turn out. This flaw has led many scholars to seek alternative, behavioral explanations.[1]

This paper re-examines these results in the presence of communication, broadly defined – between candidates, media, and voters – and shows that this can support high turnout in large elections while maintaining the assumption that voters' incentives are purely instrumental. The key difference is that communication allows for strategies such that equilibrium behavior is still optimal for each individual voter, but such that voters' turnout decisions are now correlated, rather than independent as in the standard game-theoretic analysis. That is, communication allows us to examine correlated equilibria (Aumann, 1974, 1987). These equilibria are behaviorally more plausible than Nash since they model voters' knowledge of the other voters' equilibrium strategies as a result of communication and learning, and so can apply to electorates with less than fully informed voters, like the U.S. (Bartels, 1996).

As suggested above, the forms of communication allowed in the model are very general. The only necessary condition is that the communication results in some amount of correlation in voters' decisions. As such, the model provides a very rich space in which communication can be from a few senders to many receivers – as it would be with the media or parties communicating with voters – or between a very large number of senders and receivers. In this sense, the model can provide a micro-foundation for group-based voter mobilization: as mobilization efforts induce correlation in decisions, they provide a mechanism for turnout that does not rely on group-based utilities or coercion (Uhlaner (1989), Schram and van Winden (1991), Cox (1999)). Moreover, as correlation could be induced by any signal – even signals like weather, which would not be thought of as having political content – the model incorporates mechanisms that would not play any role in standard rational choice explanations.[2]

The intuition underlying the highest-turnout correlated equilibrium is straightforward. To see this, suppose there are two parties, A and B, who compete in an election decided by majority rule. Citizens (potential voters) are not indifferent between the parties, so there are n_A citizens that support party A and $n_B < n_A$ citizens that support

[1]See, e.g., Feddersen and Sandroni (2006), Bendor et al. (2011), Ali and Lin (2013).

[2]See Gomez, Hansford, and Krause (2007) who demonstrate not only that the bad weather on the election day decreases turnout, but also that it affects Democrats and Republicans differently.

party B. Each citizen decides to vote based only on the tradeoff between her potential effect on the election outcome and the cost of voting. A voter will only affect the outcome when pivotal, that is, when her vote would change the election from her least favored party winning to a tie, or from a tie to her most favored party winning. As in standard models, in any equilibrium, the probability that a voter is pivotal, multiplied by the benefit she gets from changing the outcome of the election, must be greater than or equal to the cost of voting. Thus turnout is highest when the election results in a tie, either directly or in expectation.

Without communication, citizens will make turnout decisions independently. The largest tie would require all of minority citizens (n_B citizens), and the exact same number of majority citizens (n_B out of n_A citizens) to participate. In such a case, every recruited citizen would be pivotal with the same probability, and so, as long as it is high enough, would have incentives to turn out as required by this strategy. But the remaining $n_A - n_B$ majority citizens would deviate by also turning out, so this is not an equilibrium. In fact, except for few very special cases, there are no equilibria where all citizens use pure strategies.

With communication, however, turnout decisions can be correlated. The party supported by the minority of the citizens signals all of its supporters to vote. The party with the majority support uses a more complicated communication protocol.[3] In some fraction of elections, p, the majority party creates a pivotal situation by sending a signal to vote to n_B of its supporters and no signal to the rest of majority citizens. In the remaining fraction of elections, the majority party sends to all of its supporters a signal to vote with probability $\frac{n_B}{n_A}$, and no signal with probability $1 - \frac{n_B}{n_A}$. Therefore, each minority citizen will be pivotal with probability p. As long as p is high enough, all minority citizens will find it in their interest to turn out and vote. On the other hand, the majority citizens, based on the signal from their party, will not know for sure whether or not they are in the pivotal situation. For the value of p corresponding to the correlated equilibrium, majority citizens will also find it in their interest to follow the signal of their party, and to avoid the cost of voting by abstaining if they receive no signal. It is easy to see that in this correlated equilibrium the expected turnout will be quite high: twice the size of the minority. If the minority is large enough, voter turnout could thus be close to 100%.

The upper bound on turnout of twice the size of the minority, highlighted in the example above, is sometimes closely approached by the actual elections. To take a recent high profile elections, the 2014 referendum on Scottish independence gathered

[3] I thank the anonymous referee for suggesting the idea of this example.

1,617,989 votes in favor of independence.[4] Internet, telephone, and face-to-face opinion polls, averaged over the last two months before the referendum day indicated that about 42.07% of Scots supported independence, which translates into about 1,802,023 citizens.[5] Assuming that polls more or less perfectly revealed the majority and minority supports, this means that nearly 90% of minority citizens turned out, which is close to the full minority turnout in the example. Moreover, the total turnout was 3,619,915 citizens, which is almost exactly twice the size of the minority (up to a third decimal point).

The remainder of the paper is organized as follows. Subsection 1.1 provides a literature overview. Section 2 describes the basic model, which assumes complete information and homogenous voting costs. Subsections 2.1.1 and 2.1.2 present and discuss the main results for this case. Subsection 2.1.3 presents efficiency analysis for the basic model. Section 3 extends the basic model to the case of heterogeneous voting costs and shows that the main results continue to hold. Section 4 explores the effects of private information about voting costs. Section 5 discusses how our results extend the related findings in the existing literature. Section 6 concludes.

1.1 Related Literature

Our paper directly relates to two strands of the voluminous literature on formal models of turnout. One is the pivotal voter model, in particular, Palfrey and Rosenthal (1983, 1985). The other is group-based models that build upon the pivotal voter analysis, e.g. Morton (1991). Our model combines these approaches, and so contributes to the literatures on the turnout paradox and voter mobilization.

The turnout paradox, that is, the unsupportable rational choice prediction of turnout rate close to zero in large elections, was first formulated by Downs (1957) in the context of a decision theoretic voting model, which was extended later by Tullock (1967) and Riker and Ordeshook (1968). It would be impossible to mention here all the relevant papers that have been published on the topic since those early studies, so we have to restrict ourselves to the most closely related works. We refer the reader to Feddersen (2004) and Geys (2006) for very well-written recent literature surveys. See also Palfrey (2013) for a recent survey of laboratory experiments in political economy, including experiments testing different theories of turnout (*Ibid.*, Section 4).

The pivotal voter model of Palfrey and Rosenthal (1983) argues that voters' decisions to turn out are strategic, so the probability of being pivotal must be determined endogenously in equilibrium. Under complete information and common voting cost, Palfrey and

[4]Source: www.bbc.com/news/events/scotland-decides/results

[5]Source: whatscotlandthinks.org/questions/should-scotland-be-an-independent-country-1

Rosenthal (1983) found several classes of high-turnout Nash equilibria. Under incomplete information about voting costs, though, Palfrey and Rosenthal (1985) showed that non-zero turnout rate in large elections is not sustainable in the (quasi-symmetric) Bayesian Nash equilibrium: only voters with non-positive voting costs will vote in the limit as the majority and minority groups get large. Myerson (1998, 2000) introduced a very general approach to the analysis of large games with population uncertainty. However his "independent actions" assumption, which results in the number of players being a Poisson random variable, does not allow correlation between players' strategies. Barelli and Duggan (2013) prove existence of a pure strategy Bayesian Nash equilibrium in games with correlated types and interdependent payoffs. Their Example 2.4, an application of their main purification theorem, is a more general version of the costly voting game under incomplete information than the one we consider in Section 4. Unlike them, we study the strategic form correlated equilibria of this game that differ from Bayesian Nash equilibria with correlated types, and focus on characterizing the bounds on expected turnout rather than equilibrium existence.

Although the pivotal voter model prediction about expected turnout fails under incomplete information, the comparative static predictions are largely supported in laboratory experiments: see, e.g. Levine and Palfrey (2007). More recent work falling within this approach focused on welfare effects associated with turnout, comparison of mandatory and voluntary voting rules, and the effect of polls (e.g. Börgers (2004), Goeree and Grosser (2007), Diermeier and Van Mieghem (2008), Krasa and Polborn (2009), Taylor and Yildirim (2010)). Campbell (1999) finds that decisive minorities (i.e., those with lower voting costs or with greater expected benefits) are more likely to win in a quasi-symmetric equilibrium, even if their expected share in the electorate is small. His main point of departure from Palfrey and Rosenthal (1985) is introducing correlation between voter types (i.e., party preference) and voting cost. In this respect, he extends Ledyard (1984) who assumed that types and costs are distributed independently.

Kalandrakis (2007, 2009) looks at general turnout games with complete information and heterogeneous costs, and shows that almost all Nash equilibria of these games are regular and robust to small amounts of incomplete information. These findings can be compared to our results in Sections 3 and 4. Another closely related paper is Myatt (2012), who investigates how adding aggregate uncertainty about candidates' popularity could be used to solve the turnout paradox. His main result can be viewed as adding a modicum of correlation in an asymptotic approximation of the high-turnout quasi-symmetric Nash equilibrium characterized in Palfrey and Rosenthal (1985) to rule out zero equilibrium turnout rate as the electorate grows large. Similarly to those equilibria, it requires the common voting cost to be high enough, and predicts a tie in the equilibrium. Myatt (2012, Proposition 2) shows that the same logic can be applied to mixed-pure Nash equilibria, but characterizes the expected turnout only for a special case of the candidates' popularity density. Our results allow for correlation directly in the solution concept.

There are other prominent approaches to modeling voter behavior that aim at solving the turnout paradox (e.g., the ethical voter model of Feddersen and Sandroni (2006); see also the recent extensions by Evren (2012) and Ali and Lin (2013); or adaptive learning models, e.g., Bendor et al. (2011); or models based on uncertainty about candidates, e.g. Sanders (2001); or the quality of voters' private signals, e.g. McMurray (2013)). While these and similar models highlight a number of important aspects of voting in mass elections, they do not explicitly consider correlations in voters' actions. Our approach in this paper is different: we deliberately maintain the stark rational choice setting to show that even in this case the high turnout equilibria can be supported once correlation among voters is accounted for.[6]

Unlike the pivotal voter model, where the individual voter is a central unit of analysis, group-based models operate at the level of groups of voters. An early example is Becker (1983), who models competition among pressure groups for political influence non-strategically as independent utility maximization by each group subject to a joint budget constraint. Uhlaner (1989) emphasizes the role of groups in voting decisions, but does not characterize the equilibrium of the model. Morton (1991) shows that with fixed candidates' positions, positive turnout can be obtained in equilibrium with two groups, but in the general equilibrium framework, where candidates' positions can shift, the paradox prevails. Schram (1991) and Schram and van Winden (1991) develop a model with two groups and opinion leaders in each group, who produce social pressure on others to turn out. The individual voters are modeled as consumers of social pressure. It is shown that it is optimal for the producers of social pressure to do it, but to explain why consumers of social pressure would find it optimal to follow the leaders a civic duty argument is used. Shachar and Nalebuff (1999) develop a model of a pivotal leader, and structurally estimate it using voting data for U.S. presidential elections. See Rosenstone and Hansen (1993), Cox (1999), and references therein for an overview of empirical findings related to party mobilization models.

Overall, group-based models get around the turnout paradox by assuming the exis-

[6] The effects of communication on turnout may be also indirect. For example, Ortoleva and Snowberg (2015) find, inter alia, that voter overconfidence, even conditional on ideology, increases turnout. Communication among voters might be a possible way that overconfidence builds up in the first place.

tence of a small number of group leaders who control voter mobilization decisions by allocating resources or by means of social pressure. The exogenous mapping from mobilization efforts to voter turnout is assumed. The micro foundation for the control mechanism as well as the origins of group leaders are not usually modeled. In our case, both of these mechanisms arise naturally as coordination mechanisms in the form of pre-play communication among voters. Communication in turn induces correlation among the voters' strategies that can lead to surprisingly high turnout.

There is growing field and laboratory experimental evidence that communication among voters, and between political activists and voters, taken in a wide variety of forms (e.g., public opinion polls, get-out-the-vote campaigns, and so forth) *critically influences turnout rates*. A book-length treatment of field experiments studying effects of get-out-the-vote campaigns on turnout is Gerber and Green (2008), and one of influential earlier papers is Gerber and Green (2000). Gerber et al (2011) show that effects of TV advertising may be strong but short-lived. See also Lassen (2005) on a related topic of voter information affecting turnout.[7] Recently, DellaVigna et al (2014) emphasize the social pressure aspect of turnout, also studied in Gerber, Green, and Larimer (2008), while Barber and Imai (2014) show that even the neighborhood composition itself may matter for turnout. A recent work by Sinclair (2012) emphasizes the role of networks in political behavior, arguing that networks not only provide information, but also directly influence citizens' actions. See also Rolfe (2012). This approach is complementary to our work: while we do not explicitly model social connections among voters in this paper, one can easily imagine how such network links could serve as channels of pre-play communication.

Laboratory experiments include, e.g., Grosser and Schram (2006), who study the effects of communication in the form of neighborhood information exchange between an early voter (sender) and a late voter (receiver) from the same neighborhood. Grosser and Schram (2010), and Agranov et al. (2013) study the effects of polls on turnout and welfare. In particular, Agranov et al. (2013) show that while polls do not have negative welfare effects, they overestimate voter turnout. The authors also find evidence for voting with the winner, where a voter is more likely to turn out if she thinks her preferred candidate is more likely to win.

[7] McMurray (2012) notes that models that avoid the turnout paradox by introducing consumption benefits, at the same time nullify the empirical relation between voter information and turnout.

2 The Model

The set of voters is denoted N, with $|N| = n \geq 3$. There are two candidates, A and B. The decision making rule is simple majority with ties broken randomly. Each player $i \in N$ has type[8] $t_i \in \{A, B\}$ representing her political preference: if $t_i = A$ then i prefers candidate A to candidate B, if $t_i = B$ then the preference is reversed. Denote by N_A, with $|N_A| = n_A$, the group of voters who prefer candidate A, and N_B, with $|N_B| = n_B$, the group preferring candidate B. Throughout the paper we assume that $n_A > n_B$, and will refer to N_A and N_B as majority and minority, respectively. Thus in the usual parlance, candidate A is the *favorite*, while candidate B is the *underdog*.

Each voter has two pure actions: to vote for the preferred candidate (action 1) or abstain (action 0).[9] Thus i's action space is $S_i = \{0, 1\}$. The set of voting profiles is $S = S_1 \times \cdots \times S_n$, i.e. $S = \{(s_i)_{i \in N} | s_i \in \{0, 1\}\}$. Voting is costly, and utility of voting net of voting cost is normalized to 1 if the preferred candidate wins, 1/2, if there is a tie, and 0 otherwise. Instead of explicitly modelling candidates as players of this game, we use a representation with a centralized mediator giving out recommendations to voters, who either maximizes or minimizes total expected turnout. As will be clear from Proposition 1, our main result, this does not matter for the empirically relevant case of the large minority with $n_B > \frac{1}{2} n_A$. In Pogorelskiy (2015) we analyze the general case where this representation matters.

2.1 Complete Information and Homogenous Voting Costs

In this section we assume that N_A and N_B are commonly known. Furthermore, assume that the participation cost is the same for all voters and fixed at $c \in (0, 1/2)$.[10] In a more general case with heterogeneous costs, considered in Section 3, we discuss how one could allow *some* voters, e.g., those who view voting as a social duty, to have negative voting costs. In the case of a negative *common* cost, however, letting $c < 0$ results in a trivial equilibrium with everybody voting, so for the rest of this section we only consider non-negative values of c.

Definition 1. *A* correlated equilibrium *is a probability distribution*[11] $\mu \in \Delta(S)$ *such that*

[8] We do not explicitly include i's private voting cost in her type for convenience reasons and always refer to i's voting costs separately.

[9] Voting for a less preferred candidate is always dominated, and can be dispensed with.

[10] If $c \geq \frac{1}{2}$ ($c \leq 0$), the problem is trivial, with abstaining (voting) being everyone's dominant strategy.

[11] Aumann (1987) calls this object a *correlated equilibrium distribution*; this distinction is immaterial.

for all $i \in N$, for all $s_i \in \{0,1\}$, and all $s_i' \in \{0,1\}$

$$\sum_{s_{-i} \in S_{-i}} \mu(s_i, s_{-i}) \left(\mathcal{U}_i(s_i, s_{-i}) - \mathcal{U}_i(s_i', s_{-i})\right) \geq 0 \tag{1}$$

where $\mathcal{U}_i(s_i, s_{-i})$ is the utility of voter i at a strategy profile (s_i, s_{-i}).

To get some intuition for this definition, assume for a moment that all joint strategy profiles have a strictly positive probability, and divide both sides of (1) by $\text{Prob}(s_i) = \sum_{s_{-i} \in S_{-i}} \mu(s_i, s_{-i})$. Since $\text{Prob}(s_{-i}|s_i) = \mu(s_i, s_{-i})/\text{Prob}(s_i)$, correlated equilibrium can be interpreted as a probability distribution over joint strategy profiles where at every profile player i's choice is a weak best response under the posterior distribution conditional on that choice. Conditioning is used here to obtain the others' posteriors about player i's choice, which must be correct in equilibrium. Notice also that Nash equilibrium is a special case of correlated equilibrium, where μ is the product of n independent probability distributions, each one over the corresponding player's action space. Thus Nash equilibrium rules out any correlation between players' actions.

Call (1) voter i's incentive compatibility (IC) constraints. Since each player has only two (pure) strategies, we only need to consider those inequalities in (1) where $s_i' \neq s_i$; thus for each of n players we will only need two inequalities making it $2n$ inequalities in total (plus the feasibility constraints on μ). Denote $\mathcal{D}(N_A, N_B, c)$ the set of solutions to such a system. Formally,

$$\mathcal{D}(N_A, N_B, c) = \{\mu \in \Delta(S) | \text{ for all } i \in N, (1) \text{ holds}\} \tag{2}$$

$\mathcal{D}(N_A, N_B, c)$ is a convex compact set, and since any Nash equilibrium is a correlated equilibrium, $\mathcal{D}(N_A, N_B, c)$ is also non-empty. It will be convenient to explicitly rewrite (2) as the set of distributions $\mu \in \Delta(S)$ such that $\forall i \in N$ the following two inequalities hold

$$\sum_{s_{-i} \in S^{-i}} \mu(0, s_{-i}) \left(\mathcal{U}_i(0, s_{-i}) - \mathcal{U}_i(1, s_{-i})\right) \geq 0 \tag{3}$$
$$\sum_{s_{-i} \in S^{-i}} \mu(1, s_{-i}) \left(\mathcal{U}_i(1, s_{-i}) - \mathcal{U}_i(0, s_{-i})\right) \geq 0 \tag{4}$$

Substituting the expression for the voter's utility with normalized benefit minus voting cost, conditions (3)-(4) reduce to

$$c \sum_{s_{-i} \in V_D^i} \mu(0, s_{-i}) + \left(c - \frac{1}{2}\right) \sum_{s_{-i} \in V_P^i} \mu(0, s_{-i}) \geq 0 \tag{5}$$

$$-c \sum_{s_{-i} \in V_D^i} \mu(1, s_{-i}) + \left(\frac{1}{2} - c\right) \sum_{s_{-i} \in V_P^i} \mu(1, s_{-i}) \geq 0 \tag{6}$$

where for any $i \in N_j$, $j \in \{A, B\}$

$$V_P^i = \left\{ (s_k)_{k \in N \setminus \{i\}} | \sum_{k \in N_j \setminus \{i\}} s_k = \sum_{k \in N_{-j}} s_k \text{ or } \sum_{k \in N_j \setminus \{i\}} s_k = \sum_{k \in N_{-j}} s_k - 1 \right\} \tag{7}$$

$$V_D^i = \left\{ (s_k)_{k \in N \setminus \{i\}} | \sum_{k \in N_j \setminus \{i\}} s_k > \sum_{k \in N_{-j}} s_k \text{ or } \sum_{k \in N_j \setminus \{i\}} s_k < \sum_{k \in N_{-j}} s_k - 1 \right\} \tag{8}$$

are the sets of profiles where player i is pivotal, and not pivotal, respectively. In the latter case, we call player i a *dummy*, hence the subscript.

Conditions (5)-(6) have a simple interpretation. They say that in any correlated equilibrium, unlike in the Nash equilibrium, for each player there are two best response conditions: one, (6), is conditional on voting, and the other, (5), conditional on abstaining. These conditions are equivalent to the following two restrictions:

$$c \geq \frac{1}{2} \text{Prob}(i \text{ is pivotal} \mid i \text{ abstains})$$
$$c \leq \frac{1}{2} \text{Prob}(i \text{ is pivotal} \mid i \text{ votes})$$

Thus, a correlated equilibrium in this game is given by a probability distribution over joint voting profiles where at every profile each player finds it incentive compatible to follow her prescribed choice conditional on this profile realization.

Out of many possible correlated equilibria, we focus on the boundaries of the set: we study the equilibria that maximize (max-turnout) and minimize (min-turnout) expected turnout. Formally, a max-turnout equilibrium solves the following linear programming problem:

$$\begin{aligned} &\text{maximize } f(\mu) = \sum_{s \in S} \left(\mu(s) \sum_{i \in N} s_i \right) \\ &\text{s.t.} \qquad \mu \in \mathcal{D}(N_A, N_B, c) \end{aligned} \tag{9}$$

for $0 < c < 1/2$. Correspondingly, a min-turnout equilibrium solves

$$\text{minimize } f(\mu) \text{ s.t. } \mu \in \mathcal{D}(N_A, N_B, c) \tag{10}$$

A potential difficulty in deriving the analytical solution to these problems lies in the $2n$ incentive compatibility constraints (5)-(6) that must be simultaneously satisfied. Fortunately, it is possible to overcome this problem. The simplification comes from the observation that for all correlated equilibria that maximize or minimize turnout, there exists a "group-symmetric" probability distribution that delivers the same expected turnout.

Let $\mu(z_i, a, b)$ denote the probability of any joint profile where player i plays strategy z_i, and, among the other $n-1$ players, a players turn out in group N_A and b players turn out in group N_B. Define a set of group-symmetric probability distributions as follows.

$$\begin{aligned}\mathcal{M} = \{\mu \in \mathcal{D}(N_A, N_B, c)|\\ &\forall i \in N_A, \forall a \in \{1, \dots, n_A - 1\}, \forall b \in \{0, \dots, n_B\} : \mu(0_i, a, b) = \mu(1_i, a-1, b)\\ &\forall k \in N_B, \forall b \in \{1, \dots, n_B - 1\}, \forall a \in \{0, \dots, n_A\} : \mu(0_k, a, b) = \mu(1_k, a, b-1)\}\end{aligned}$$

In words, the distributions in $\mathcal{M}$ place the same probability on all such profiles that have the same number of players turning out from either side, and differ only by the identity of those who turn out and those who abstain. Thus the identity of the voter does not matter as long as the total number of this voter's group votes is the same, given the fixed number of votes on the other side.

Lemma 1. *For any distribution $\mu^* \in \mathcal{D}(N_A, N_B, c)$ that solves problem (9) or (10), there exists an equivalent group-symmetric probability distribution σ^* that also delivers a solution to the same problem. Formally, $f(\sigma^*) = f(\mu^*)$ and $\sigma^* \in \mathcal{M}$.*

Proof. See A.1. □

Lemma 1 allows a substantial simplification of the problem without any loss of generality, reducing $2n$ inequalities down to just four: two for a member of group N_A and two more for a member of group N_B; and reducing the number of variables (unknown profile probabilities) from the original 2^n profiles down to $(n_A+1)(n_B+1)$, which is the maximal number of profiles with different probabilities under group-symmetric distributions.

Before describing the general characterization of solutions to (9) and (10), we walk through the simplest possible example with 3 voters, which serves to illustrate both Lemma 1 and the main results of the paper.

Example 1. Suppose $N = \{1, 2, 3\}$. Let $N_A = \{1, 2\}$ and $N_B = \{3\}$. There are eight possible voting profiles: from $(0, 0, 0)$ with no one voting to $(1, 1, 1)$ with full turnout. Denote (s_i, s_j, s_k) a strategy profile where $i, j \in N_A$ and $k \in N_B$. Then for each $i \in N_A$,

$$\mathcal{U}_i(s_i, s_{-i}) = \begin{cases} 1 - s_i c & \textit{if } (s_i, s_j, s_k) \in \{(0,1,0), (1,0,0), (1,1,0), (1,1,1)\} \\ \frac{1}{2} - s_i c & \textit{if } (s_i, s_j, s_k) \in \{(0,0,0), (0,1,1), (1,0,1)\} \\ 0 & \textit{if } (s_i, s_j, s_k) = (0,0,1) \end{cases}$$

Similarly, for $k \in N_B$,

$$\mathcal{U}_k(s_k, s_{-k}) = \begin{cases} 1 - c & \textit{if } (s_i, s_j, s_k) = (0,0,1) \\ \frac{1}{2} - s_k c & \textit{if } (s_i, s_j, s_k) \in \{(0,0,0), (0,1,1), (1,0,1)\} \\ -s_k c & \textit{if } (s_i, s_j, s_k) \in \{(0,1,0), (1,0,0), (1,1,0), (1,1,1)\} \end{cases}$$

Denote $\mu_{s_i s_j s_k} = \mu(s_i, s_j, s_k)$ to simplify notation. Now conditions (5)-(6) reduce to the following system of linear inequalities, where we also add the standard probability requirements:

$$c\mu_{010} + \left(c - \frac{1}{2}\right)(\mu_{000} + \mu_{001} + \mu_{011}) \geq 0 \tag{11}$$

$$-c\mu_{110} + \left(\frac{1}{2} - c\right)(\mu_{100} + \mu_{101} + \mu_{111}) \geq 0 \tag{12}$$

$$c\mu_{100} + \left(c - \frac{1}{2}\right)(\mu_{000} + \mu_{001} + \mu_{101}) \geq 0 \tag{13}$$

$$-c\mu_{110} + \left(\frac{1}{2} - c\right)(\mu_{010} + \mu_{011} + \mu_{111}) \geq 0 \tag{14}$$

$$c\mu_{110} + \left(c - \frac{1}{2}\right)(\mu_{000} + \mu_{010} + \mu_{100}) \geq 0 \tag{15}$$

$$-c\mu_{111} + \left(\frac{1}{2} - c\right)(\mu_{001} + \mu_{011} + \mu_{101}) \geq 0 \tag{16}$$

$$\forall s \in \{0,1\}^3 \quad \mu_s \geq 0 \tag{17}$$

$$\sum_{s \in \{0,1\}^3} \mu_s = 1 \tag{18}$$

The solutions have the following properties.[12] In any correlated equilibrium the con-

[12] Recall that we restricted c to lie in (0,0.5). We can now provide the rationale behind this assumption. If $c > 0.5$, the unique correlated equilibrium has $\mu_{000} = 1$, i.e., no one votes. This follows because once $c > \frac{1}{2}$, inequalities (12), (14), and (16) can only hold if $\mu_{100} = \mu_{101} = \mu_{110} = \mu_{111} = 0$, $\mu_{010} = \mu_{011} = 0$, and $\mu_{001} = 0$, which implies $\mu_{000} = 1$. If $c = 0.5$, any probability distribution with $\mu_{110} = 0$ and $\mu_{111} = 0$ is a correlated equilibrium: inequalities (12), (14), and (16) can only hold if $\mu_{110} = \mu_{111} = 0$, while all remaining inequalities are trivially satisfied. If $c = 0$, then any probability distribution with

straints can be rewritten as

$$\frac{\frac{1}{2}-c}{c}(\mu_{000}+\mu_{010}+\mu_{100}) \leq \mu_{110} \leq \frac{\frac{1}{2}-c}{c}\left(\mu_{111}+\min\left\{\mu_{100}+\mu_{101},\mu_{010}+\mu_{011}\right\}\right) \tag{19}$$

$$\mu_{010} \geq \frac{\frac{1}{2}-c}{c}(\mu_{000}+\mu_{001}+\mu_{011}) \tag{20}$$

$$\mu_{100} \geq \frac{\frac{1}{2}-c}{c}(\mu_{000}+\mu_{001}+\mu_{101}) \tag{21}$$

$$\mu_{111} \leq \frac{\frac{1}{2}-c}{c}(\mu_{001}+\mu_{011}+\mu_{101}) \tag{22}$$

$$\sum_{s\in\{0,1\}^3} \mu_s = 1 \tag{23}$$

$$\mu_{000},\mu_{001},\mu_{010},\mu_{011},\mu_{100},\mu_{101},\mu_{111} \in [0,1), \mu_{110} \in (0,1) \tag{24}$$

This system has many solutions, and $\mu_{000} < 1$ implies that all have positive expected turnout. Notice that in (20)-(22) the probabilities of profiles with more votes are bounded from above by the probabilities of profiles with less votes, while in (19) it is the other way round. These relations are important for the extreme correlated equilibria, because they determine the constraints that bind at an optimum.

We next identify the max-turnout equilibria that solve the following linear program:

$$\text{maximize} \sum_{s\in\{0,1\}^3}(s_i+s_j+s_k)\mu_{s_is_js_k} \quad \text{s.t.} \quad \mu \in \mathcal{D}(2,1,c) \tag{25}$$

A solution to (25) always exists since $\mathcal{D}(2,1,c) \neq \emptyset$. We will denote such a solution μ^*. Since the objective function does not depend on $\mu_{000} \geq 0$, (23) implies that $\mu^*_{000} = 0$. Using this fact and (23), we can rewrite the objective in (25) as

$$\sum_{s\in\{0,1\}^3}(s_i+s_j+s_k)\mu_{s_is_js_k} = 1 + (\mu_{011}+\mu_{101}+\mu_{110}) + 2\mu_{111} \tag{26}$$

We next show that at μ^* the value of the objective function is 2 for any $0 < c < 0.5$. Lemma 1 implies that without loss of generality we can let $\mu_{010} = \mu_{100}$ and $\mu_{011} = \mu_{101}$.

$\mu_{000} = \mu_{001} = \mu_{011} = \mu_{101} = \mu_{010} = \mu_{100} = 0$ is a correlated equilibrium; thus it is any mixture between μ_{111} and μ_{110}.

Hence (20) and (21) reduce to the same constraint, and (19)-(23) imply[13]

$$
\begin{aligned}
\mu_{010} &\leq \mu_{111} + \mu_{011} \\
\mu_{010} &\geq \frac{\frac{1}{2}-c}{c}(\mu_{001} + \mu_{011}) \\
\mu_{111} &\leq \frac{\frac{1}{2}-c}{c}(\mu_{001} + 2\mu_{011}) \\
\sum_{s\in\{0,1\}^3} \mu_s &= 1
\end{aligned}
$$

where the first inequality follows from (19) with $\mu^*_{000} = 0$. This implies

$$\mu_{111} \leq 2\mu_{010} - \frac{\frac{1}{2}-c}{c}\mu_{001}$$

Then in (26) the right hand side is at most $1 + (\mu_{011} + \mu_{101} + \mu_{110} + \mu_{010} + \mu_{100} + \mu_{111}) - \frac{\frac{1}{2}-c}{c}\mu_{001} = 2 - \frac{\mu_{001}}{2c}$. Now we can see that to achieve the upper bound of two, it is necessary to put $\mu^*_{001} = 0$. Thus we let $\mu^*_{000} = \mu^*_{001} = 0$, and put $\mu^*_{111} = 2\mu^*_{010}$. Then constraints (19)-(23) reduce to

$$
\begin{aligned}
\frac{\frac{1}{2}-c}{c}2\mu^*_{010} \leq \mu^*_{110} &\leq \frac{\frac{1}{2}-c}{c}\left(3\mu^*_{010} + \mu^*_{011}\right) &(27)\\
\mu^*_{010} &\geq \frac{\frac{1}{2}-c}{c}\mu^*_{011} &(28)\\
2\mu^*_{010} &\leq \frac{\frac{1}{2}-c}{c}2\mu^*_{011} &(29)\\
\sum_{s\in\{0,1\}^3} \mu^*_s &= 1 &(30)
\end{aligned}
$$

From the last two inequalities it follows that $\mu^*_{011} = \mu^*_{101} = \frac{c}{\frac{1}{2}-c}\mu^*_{010}$. Re-arranging,

$$
\begin{aligned}
\frac{\frac{1}{2}-c}{c}2\mu^*_{010} \leq \mu^*_{110} &\leq \frac{\frac{1}{2}-c}{c}\left(3 + \frac{c}{\frac{1}{2}-c}\right)\mu^*_{010} &(31)\\
\mu^*_{011} &= \frac{c}{\frac{1}{2}-c}\mu^*_{010} &(32)\\
2\mu^*_{011} + \mu^*_{110} + 4\mu^*_{010} &= 1 &(33)
\end{aligned}
$$

Replacing $\mu^*_{110} = 1 - \frac{1-c}{1/2-c}2\mu^*_{010}$ from (33) and re-arranging, we obtain the following

[13] For the sake of brevity, we omit the non-negativity constraints on μ.

system, which, if holds, delivers the value of two to the objective function:

$$\begin{aligned}
\mu^*_{000} &= \mu^*_{001} = 0 \\
\mu^*_{111} &= 2\mu^*_{010} \\
\mu^*_{011} &= \frac{c}{\frac{1}{2}-c}\mu^*_{010} \\
\mu^*_{110} &= 1 - 2\frac{1-c}{\frac{1}{2}-c}\mu^*_{010} \\
\frac{\frac{1}{2}-c}{c}2\mu^*_{010} \leq 1 - 2\frac{1-c}{\frac{1}{2}-c}\mu^*_{010} &\leq \frac{\frac{1}{2}-c}{c}\left(3+\frac{c}{\frac{1}{2}-c}\right)\mu^*_{010}
\end{aligned}$$

This system has at least one solution for all $c \in (0, 0.5)$. In particular, we can put

$$\begin{aligned}
\mu^*_{000} &= \mu^*_{001} = 0 \\
\mu^*_{010} &= \mu^*_{100} = c(1-2c) \\
\mu^*_{111} &= 2c(1-2c) \\
\mu^*_{011} &= \mu^*_{101} = 2c^2 \\
\mu^*_{110} &= 4c^2 - 4c + 1
\end{aligned}$$

One can verify that for this distribution, all original constraints hold, and the value of the objective function is two. Hence for any cost $0 < c < 0.5$, we can find a correlated equilibrium with expected turnout being exactly two out of three voters, i.e. twice the size of the minority. We will see shortly that this is a general property of the max-turnout correlated equilibria.

2.1.1 Max-turnout equilibria

Let us now turn to the general case. Recall that we want to solve the following problem for $0 < c < 1/2$:

$$\begin{aligned}
&\text{maximize } f(\mu) = \sum_{s\in\{0,1\}^n}\left(\mu(s)\sum_{i\in N} s_i\right) \\
&\text{s.t.} \qquad \mu \in \mathcal{D}(N_A, N_B, c)
\end{aligned} \tag{34}$$

Let $f^* \equiv f(\mu^*)$ be the value of the objective at the optimum in (34). Our first main result is the analytic solution to the max-turnout problem for all costs in the specified range.

Proposition 1. *Suppose* $0 < c < 0.5$*,* $n_A, n_B \geq 1$*, and* $n_A > n_B$*. Then the following*[14]

[14] In terms of notation, $\lceil x \rceil$ stands for the smallest integer not less than x.

holds:

(i) if $n_B \geq \lceil \frac{1}{2} n_A \rceil$, *then* $f^* = 2n_B$;

(ii) if $n_B < \lceil \frac{1}{2} n_A \rceil$, *then*

$$f^* = 2n_B + \frac{(n_A - 2n_B)(1 - 2c)}{1 + 2c\left(\frac{n_A(n_A-1)}{n_A+n_B(n_A-1)} - 1\right)} = 2n_B + \phi(c),$$

where $\phi(c) \in (0, n_A - 2n_B)$ *and is decreasing in* c. *Alternatively,* f^* *can be expressed as*

$$f^* = n_A \times \frac{2cn_B(n_A - 1) + n_B(n_A - 1) + n_A(1 - 2c)}{2c(n_A - n_B)(n_A - 1) + n_B(n_A - 1) + n_A(1 - 2c)} = n_A \times \xi(c)$$

where $\xi(c)$ *is decreasing in* c, *and*

a) $\xi(c) \in (0, 1)$ *for all* $0 < c < \frac{1}{2}$;

b) $\xi(c) \to \frac{2n_B}{n_A}$ *as* $c \to \frac{1}{2}$, *so* $f^* \to 2n_B$;

c) $\xi(c) \to 1$ *as* $c \to 0$, *so* $f^* \to n_A$.

Remark 1. The proof of Proposition 1 is in Appendix A.2. Lemma 1 is fundamental in proving this result, allowing to establish the optimum and characterize the max-turnout equilibrium support under a group-symmetric distribution (see Corollary 1 below). The intuition for the result is as follows. To maximize turnout, the largest probability mass must be placed on the voting profile where everyone votes. However, since $n_A > n_B$, the *voting* players from N_B are not pivotal at this profile, so for those players constraint (6) binds at the optimum. This implies that constraint (5) for *abstaining* players in N_A binds at the optimum, because from (6) for players in N_B binding, the probability of the largest profile can be expressed via the probabilities of profiles where the *voting* players from N_B are pivotal, and those are precisely the profiles where *abstaining* players from N_A are pivotal. The key difference between cases (i) and (ii) only concerns the behavior of constraint (5) for players in N_B and constraint (6) for players in N_A. Using these binding constraints and the total probability constraint allows us to get a constructive characterization of the optimum.

Proposition 1 shows that all max-turnout correlated equilibria exhibit a substantial turnout of at least $2n_B$ for all common costs in the range where neither voting nor abstention is a dominant strategy, and for groups of different sizes. Max-turnout equilibria have a very natural interpretation: when the group sizes are so different that the minority have a priori low chances of winning even when the majority group votes at random (i.e., $n_B < \lceil \frac{1}{2} n_A \rceil$), the cost of voting matters and the maximal expected turnout is *decreasing* in cost. When the group size difference is not that large, the maximal expected turnout equals twice the size of the minority and does not depend on cost, as if voting was costless.

In addition to the maximal expected turnout, we also characterize the support of the optimal group-symmetric distributions. Using Lemma 1, we can, without loss of generality, describe the profiles in the support as (a, b) where a (b) is the total number of voters from N_A (N_B, respectively) who turn out at this profile.

Corollary 1. *A correlated equilibrium with maximal expected turnout can be implemented via a group-symmetric distribution with the following support $\tilde{S} \subset S$:*

(i) if $n_B \geq \lceil \frac{n_A+1}{2} \rceil$, then

$$\tilde{S} = \left\{(a, n_B) \in \mathbb{Z}^2 | a \in \{0, \ldots, n_B - 2\} \cup \{n_B, \ldots, n_A\}\right\};$$

(ii) if $n_B < \lceil \frac{1}{2} n_A \rceil$, then

$$\tilde{S} = \left\{(a+1, a) \in \mathbb{Z}^2 | a \in \{0, \ldots, n_B\}\right\} \cup \{(n_B, n_B)\} \cup \{(n_A, 0)\}$$

Proof. See A.2. □

In words, when $n_B > \lceil \frac{1}{2} n_A \rceil$, the equilibrium support consists of everyone in the minority voting except at the profile $(n_B - 1, n_B)$, and the majority mixing between all profiles. When $n_B < \lceil \frac{1}{2} n_A \rceil$, the support consists only of the profiles where the minority has exactly one vote less than the majority, the largest tied profile, and a single extreme profile with the full turnout by the majority and full abstention by the minority, $(n_A, 0)$.

Group-symmetric distributions allow to characterize the correlated equilibria with maximal expected turnout without loss of generality, but this characterization is not unique: it is possible that an asymmetric probability distribution also delivers a solution to the max-turnout problem. However, the group-symmetric distribution has an attractive implementation property: all voters in a group are treated equally. Namely, one way to think about a group-symmetric correlated equilibrium is to imagine a mediator selecting a profile with a given total number of votes on each side according to the group-symmetric equilibrium distribution, μ^*, and then randomly recruiting the required number of voters on each side according to the selected profile, giving a recommendation to vote to those selected, and a recommendation to abstain to the rest. Thus the group-symmetric max-turnout equilibria involve interim randomization on the part of the mediator.

Remark 2. Based on the profiles that have positive probability in equilibrium, it is instructive to compare the correlated equilibria identified in case (i) with the mixed-pure

Nash equilibria of Palfrey and Rosenthal (1983): indeed, according to Corollary 1, just like in those equilibria, voters in N_B should vote for sure, and voters in N_A should mix. The similarity ends here, however. First, the max turnout mixed-pure Nash equilibria have expected turnout increasing in the cost. Second, in the mixed-pure equilibria of Palfrey and Rosenthal, all voters of the mixing group vote with the same probability $q \in (0,1)$. Hence the probability of a profile (a, n_B) is $\binom{n_A}{a} q^a (1-q)^{n_A - a}$. In the correlated equilibria from case (i), the probability of the same profile is $\binom{n_A}{a}\mu_{a,n_B}$, where μ delivers a maximum to the objective in (34). For the two probability distributions to coincide, it requires $\mu_{a,n_B} = q^a(1-q)^{n_A-a}$ for all $a \in [0, n_A]$. But since $\binom{n_A}{n_B}\mu_{n_B,n_B} = 2c$ (see Corollary 2 below) and $\mu_{n_B-1,n_B} = 0$, there is no $q \in (0,1)$ that would satisfy this condition.

Remark 3. If one restricts the equilibrium support in case (i) to the following three profiles: full turnout, largest tie, and any single profile of the form (a, n_B) for $a \in \{0, \ldots, n_B - 2\}$, the group-symmetric max-turnout equilibrium is unique. This follows from equations (70) and (72) in A.2. Our example in the introduction is a special case of this restricted equilibrium support with $a = 0$.

In view of Corollary 1, we can compute the probability that the election results in a tie, denoted π_{n_B,n_B}, since (n_B, n_B) is the only tied profile in the support of the equilibrium distribution. It is also interesting to see how the probability of the tie changes with the size of the electorate. There are several ways to model the limiting case when the electorate grows large. We present here the results for the simplest case, which is keeping the ratio $\frac{n_B}{n_A} = \alpha$ fixed at some $\alpha \in (0,1]$ as $n_B, n_A \to \infty$.

Corollary 2. *(i) if $n_B \geq \lceil \frac{n_A+1}{2} \rceil$, then*

$$\pi_{n_B,n_B} = 2c$$

(ii) if $n_B < \lceil \frac{1}{2} n_A \rceil$, then

$$\pi_{n_B,n_B} = \frac{2c}{1 + \left(\frac{1}{2c} - 1\right)\left(\frac{1}{n_A - 1} + \frac{n_B}{n_A}\right)}$$

(iii) for any fixed c, as $n_A, n_B \to \infty$ with $\frac{n_B}{n_A} = \alpha \in (0,1)$, for $\alpha \in (0, 0.5)$ we have $\pi_{n_B,n_B} \to \frac{2c}{1+\alpha\left(\frac{1}{2c}-1\right)}$, and for $\alpha \in (0.5, 1)$, $\pi_{n_B,n_B} \to 2c$.

Proof. See equations (71) and (93) in A.2. □

Corollary 2 shows that the probability of the tied outcome only depends on the cost and the relative size of the competing groups, and is increasing in the cost. There is one caveat: the tie probability is derived under the assumption of a group-symmetric probability distribution. For an asymmetric probability distribution that also delivers a solution to the max-turnout problem, Corollary 2 holds as long as the equilibrium support stays the same.

Another important proprety concerns the probability that the majority wins. Given Corollaries 1 and 2, it is not surprising that there are again two cases for the max-turnout equilibria:

Corollary 3. *The probability the majority wins in a correlated equilibrium with maximal expected turnout, π_m, is restricted as follows.*

(i) if $n_B \geq \lceil \frac{n_A+1}{2} \rceil$, then

$$1 - c \geq \pi_m > \frac{1}{2}$$

For the special case in Remark 3,

$$\pi_m = \frac{n_B}{n_A} + c\left(1 - \frac{2n_B}{n_A}\right)$$

(ii) if $n_B < \lceil \frac{1}{2} n_A \rceil$, then

$$\pi_m = 1 - \frac{c}{1 + \left(\frac{1}{2c} - 1\right)\left(\frac{1}{n_A - 1} + \frac{n_B}{n_A}\right)}$$

Proof. See A.3. □

Corollary 3 shows that the probability that majority wins is decreasing in the cost for a small minority (case (ii)). As $c \to 0.5$, $\pi_m \to 0.5$ from above. Furthermore, for all costs in $(0, 0.5)$ the majority wins with probability at least 0.5. In case (i), when $n_B \geq \lceil \frac{n_A+1}{2} \rceil$, the upper bound on this probability is decreasing in the cost, but the situation is a bit more complicated, since π_m is non-monotone in the cost for a fixed pair of groups sizes n_A and n_B. The reason is the non-monotone behavior of the binomial coefficients as well as the sensitivity of the linear program to the changes in the constraint coefficients. The total probability mass fluctuates along the profiles of the form (a, n_B) for $a \in \{0, \ldots, n_B - 2\} \cup \{n_B, \ldots, n_A\}$ depending on the cost, and so does the probability of the majority winning.

Our next proposition shows that as the size of the electorate grows large, the max-turnout correlated equilibria remain divided into the same two categories: the cost-independent case with the maximal expected turnout being twice the size of the minority, and the cost-dependent case, where the maximal expected turnout includes an additional term.

Proposition 2. *Fix $c \in (0, 0.5)$ and let $n_A, n_B \to \infty$ with $\frac{n_B}{n_A} = \alpha \in (0, 1]$.*

(i) If $\alpha \geq 0.5$, then

$$\lim_{n_A, n_B \to \infty} \frac{f^*}{n} = \frac{2\alpha}{1 + \alpha}$$

(ii) If $\alpha < 0.5$, then

$$\lim_{n_A, n_B \to \infty} \frac{f^*}{n} = \frac{2\alpha}{1 + \alpha} + \frac{(1 - 2\alpha)(1 - 2c)}{(1 + \alpha)(1 - 2c\left(1 - \frac{1}{\alpha}\right))}$$

Proof. See A.4. □

2.1.2 Min-turnout equilibria

Concluding the section on the basic model, let us briefly address the lower bound on the expected turnout. This case is different in that now we are looking for a solution that minimizes the linear objective function subject to the same constraints (5)-(6).

Denote the minimal expected turnout in this problem by

$$f_* \equiv f(\mu_*) = \min_{\mu \in \mathcal{D}(N_A, N_B, c)} \sum_{s \in \{0,1\}^n} \left(\mu(s) \sum_{i \in N} s_i \right) \tag{35}$$

Proposition 3. *Suppose $0 < c < 0.5$, and $n_A, n_B \geq 1$. Then $f_* = 2 - \psi(c)$, where $\psi(c) \in (0, 2)$.*

Proof. See A.5. □

As Proposition 3 shows, the lower turnout bound is not very interesting. For all cases, the minimal expected turnout is between 0 and 2, depending on the cost, and the exact formula for $\psi(c)$ is complicated, since, unlike the maximum case, the equilibrium distribution support also depends on the cost, as shown in the Appendix. On the other hand, the result is intuitive: the minimum turnout case is total cost-minimizing, so to remove the individual incentives to turn out it is sufficient to have the equilibrium distribution place all the probability mass onto the uncontested profiles where either side wins for sure. Such profiles need no more than two agents voting.[15]

[15]There is an exception to this rule when the voting cost is approaching zero, but even if profiles with total turnout larger than 2 have positive probabilities in equilibrium, their effect on the objective is completely compensated by the profiles with turnout between 0 and 2. See A.5 for details.

2.1.3 Correlated Equilibria and Efficiency

In this section we rely on the results we have obtained in the basic model to draw some general implications about the effects of correlated strategies on welfare.

Firstly, we note that since the set of expected correlated equilibrium payoffs is convex, there is always an equilibrium with the total expected turnout between the minimum and the maximum.

Proposition 4. *For any $0 < c < 0.5$ and $t \in [f_*(c), f^*(c)]$, there exists a correlated equilibrium with the total expected turnout equal to t.*

Proof. See A.6. □

Next, we ask which correlated equilibria are socially optimal. That is, we are looking for equilibria that maximize expected social welfare, understood as a sum of all individuals' expected utilities. Given a correlated equilibrium μ, after some simple algebra, the expected welfare can be formally written as follows.

$$W(\mu) = (n_A - n_B)\Pr(\textit{Majority wins}) + n_B - cT(\mu) \tag{36}$$

where $T(\mu)$ is the total expected turnout under μ. The expression in (36) nicely demonstrates the relation between total expected turnout and welfare: increasing total turnout reduces welfare if the probability that majority wins is kept constant, but it may increase welfare if the increased turnout leads to a higher probability that majority wins.

Given our results on max turnout equilibria in Section 2.1.1, we can now establish some welfare properties of such equilibria.

Proposition 5. *Suppose $0 < c < 0.5$ and $n_A > n_B$. Denote W^* the expected welfare at a max turnout equilibrium.*

(i) if $n_B \geq \lceil \frac{n_A+1}{2} \rceil$, then $W^* = (n_A - n_B)\Pr(\textit{Majority wins}) + n_B(1-2c)$; and $\frac{n_A+n_B}{2} - 2cn_B < W^* \leq n_A - c(n_A + n_B)$;

(ii) if $n_B < \lceil \frac{1}{2} n_A \rceil$, then

$$W^* = n_A(1-c)\left(1 + \frac{2cn_B(n_A-1)}{2c(n_A-n_B)(n_A-1) + n_B(n_A-1) + n_A(1-2c)}\right)$$

(iii) In both cases, W^* is decreasing in the voting cost

Correlated equilibria that maximize total welfare have lower expected turnout than the max-turnout equilibria. A welfare maximizing correlated equilibrium would require the probability that majority wins as large as possible (ideally, equal to 1) and turnout as low as possible (ideally, 0). In this case the maximum welfare equals n_A. However, there is a tradeoff between the probability majority wins and the expected turnout: majority cannot win for sure in any correlated equilibrium.

Lemma 2. *For any $0 < c < \frac{1}{2}$, there does not exist a correlated equilibrium with majority winning for sure.*

Proof. See A.7. □

Remark 4. It is interesting to note that if voting costs are different in different groups, it *is* possible to have a correlated equilibrium with majority winning for sure. In particular, if there are two group costs, c_A and c_B, then for $c_A < c_B$ both IC constraints for voters in N_A and non-voters in N_B can be satisfied. The welfare-maximizing equilibria in such case have the probability majority wins equal to one, and all probability mass on the profiles with one and two voters from N_A and zero voters from N_B.

When looking for a welfare-maximizing correlated equilibrium, Lemma 2 implies that the probability majority wins enters (36) non-trivially and must be traded off with the total expected turnout. Similarly to Lemma 1, there is no loss of generality involved from considering only group-symmetric probability distributions. We can now establish the equilibrium support for welfare-maximizing equilibria, and characterize the optimum. Formally, the problem is now

$$\text{maximize } W(\mu) \text{ s.t. } \mu \in \mathcal{D}(N_A, N_B, c) \tag{37}$$

Proposition 6. *Assume $n_A > 2$.*

i) There is a unique cutoff cost c_* such that for any $0 < c < c_*$ the maximal expected welfare implementable in a correlated equilibrium is

$$W(\mu^*, c) = n_A - c + \frac{\left[c - \frac{n_A + n_B + 2n_B(\frac{1}{2} - c)}{2} - \frac{\left(\frac{1}{2} - c\right)^2(1 + n_B)}{c}\right]}{\frac{(c + \frac{1}{2}(n_B + 1))(\frac{1}{2} - c)}{c^2} + \frac{n_B}{2c} + 1}$$

and the corresponding equilibrium support profiles are $(a+1, a), a \in [0, n_B]$, (n_B, n_B), and $(2, 0)$.

ii) for $c > c_*$ such that Condition A (see below) holds, the maximal expected welfare implementable in a correlated equilibrium is

$$\tilde{W}(\mu^*, c) = n_A - c + \frac{\left[c(1+n_B) + n_B\left[n_B - n_A - 1\right] - \frac{\left(\frac{1}{2}-c\right)^2(1+n_B)}{c}\right]}{\frac{n_B-(c+\frac{1}{2})}{\frac{1}{2}-c} + \frac{(c+\frac{1}{2}(n_B+1))(\frac{1}{2}-c)}{c^2}}$$

and the corresponding equilibrium support profiles are $(a+1, a), a \in [0, n_B]$*,* $(0,1)$*, and* $(2,0)$*.*

iii) for $c > c_*$ *such that Condition A does not hold, the maximal expected welfare implementable in a correlated equilibrium is*

$$\bar{W}(\mu^*, c) = n_A - c + \frac{(\frac{1}{2} - c)\left[n_B(n_B - n_A) - c(n_B - 1)\right]}{n_B - \left(c + \frac{1}{2}\right)}$$

and the corresponding equilibrium support profiles are $(0,1), (1,0)$*, and* $(2,0)$*.*

Proof. See A.8 □

Remark 5. The unique cutoff cost c_* is determined by equation (113) in the proof. **Condition A** in the statement of Proposition 6 is the following cubic inequality in the voting cost:

$$c^3\left(n_A + \frac{n_B - 5}{2}\right) + \frac{c^2}{2}\left((n_A - n_B)(n_B - 1) + 3 - n_B\right) \\ - \frac{c}{4}\left(\frac{n_B + 1}{2} + (n_A - n_B)(2n_B + 1)\right) + \frac{(n_A - n_B)(n_B + 1)}{8} > 0$$

This inequality is equivalent to having $\tilde{W}(\mu^*, c) > \bar{W}(\mu^*, c)$.

Proposition 6 characterizes welfare-optimal equilibria and shows that those are generally different from either min- or max-turnout equilibria, although the expected turnout in welfare-maximizing case is close to the minimal expected turnout.

3 Complete Information and Heterogeneous Voting Costs

We have assumed so far that the cost of voting is common for all players. This assumption may seem too strong, so in this section we are going to relax it and see if the main results continue to hold.

Assume that each voter $i \in N$ has a voting cost $c_i \in (0, 0.5)$ and the costs are commonly known. In this cost range, no voter has a dominant strategy to always vote or always abstain. The correlated equilibrium conditions (5)-(6) now take the following form: $\forall i \in N$,

$$c_i \sum_{s_{-i} \in V_D^i} \mu(0, s_{-i}) + \left(c_i - \frac{1}{2}\right) \sum_{s_{-i} \in V_P^i} \mu(0, s_{-i}) \geq 0 \tag{38}$$

$$-c_i \sum_{s_{-i} \in V_D^i} \mu(1, s_{-i}) + \left(\frac{1}{2} - c_i\right) \sum_{s_{-i} \in V_P^i} \mu(1, s_{-i}) \geq 0 \tag{39}$$

where, as before, V_P^i (V_D^i) is the set of voting profiles where player i is a pivotal(dummy, respectively). Denote $\mathcal{D}(N_A, N_B, (c_i)_{i \in N})$ the set of probability distributions over $\Delta(S)$ that satisfy (38)-(39).

With heterogeneous costs, the group-symmetric distribution construction (see Lemma 1), may entail some loss of generality. Since voting costs are different, the expected turnout can be increased, compared to the group-symmetric case, if the probability distribution over profiles is adjusted so that each profile probability takes into account not only the total number of those players voting at this profile, but also their voting costs. E.g., profiles where players with higher costs are voting might be optimally assigned smaller probability than profiles with the same total turnout, but where players with lower costs are voting.[16]

Without loss of generality, let us order all players in group N_A (N_B, respectively) by their voting costs from low to high. Denote $\underline{c}_A$, $\underline{c}_B$ the lowest costs in the respective groups. Similarly, denote $\bar{c}_A$, $\bar{c}_B$ the highest costs. A joint cost profile $\boldsymbol{c}_{[\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B]}$ is any cost assignment $(c_i)_{i \in N}$ to the players in N such that $\forall i \in N_j$, $j \in \{A, B\}$, $\underline{c}_j \leq c_i \leq \bar{c}_j$. Denote the maximal expected turnout in the turnout problem with heterogeneous costs by

$$h^* \equiv f(\mu^*) = \max_{\mu \in \mathcal{D}(N_A, N_B, (c_i)_{i \in N})} \sum_{s \in \{0,1\}^n} \left(\mu(s) \sum_{i \in N} s_i\right) \tag{40}$$

In the present version of the paper, we restrict our analysis to the case of symmetric distributions and demonstrate that our results under homogenous costs can be replicated as a special case. The main goal of this exercise is to show that the maximal expected turnout remains at high levels under heterogeneous costs, even if the set of admissible probability distributions is restricted to be symmetric.

[16] Nevertheless, there is an important special case with two common group costs, c_A and c_B, where one can prove an analogue of Lemma 1. We do not analyze it here.

3.1 Symmetric distributions

In this subsection, we require the probability distributions to be group-symmetric. Analogously to Lemma 1, define

$$\begin{aligned}\mathcal{M}_H := \{\mu \in \mathcal{D}(N_A, N_B, (c_i)_{i\in N})| \\ \forall i \in N_A, \forall b \in \{0,\dots,n_B\}, \forall a \in \{1,\dots,n_A-1\} : \mu(0_i, a, b) = \mu(1_i, a-1, b) \\ \forall k \in N_B, \forall b \in \{1,\dots,n_B-1\}, \forall a \in \{0,\dots,n_A\} : \mu(0_k, a, b) = \mu(1_k, a, b-1)\}\end{aligned}$$

In words, $\mathcal{M}_H$ is the set of group-symmetric probability distributions over joint profiles which are also correlated equilibria for complete information and heterogeneous costs. Denote the maximal expected turnout in the turnout problem with heterogeneous costs and group-symmetric distributions by

$$\tilde{h}^* := \max_{\mu \in \mathcal{M}_H} \sum_{s\in\{0,1\}^n} \left(\mu(s) \sum_{i\in N} s_i\right) \tag{41}$$

Clearly, $h^* \geq \tilde{h}^*$. We will now show that an analogue of Proposition 1 holds under the condition $\underline{c}_A = \bar{c}_B$.

Proposition 7. *Suppose $0 < c_i < 0.5$ for all $i \in N$. Require $\mu \in \mathcal{M}_H$. Then the following expressions for $\tilde{h}^*$ provide the optimal value to the objective in the max turnout problem with heterogeneous costs and group-symmetric distributions if and only if $\underline{c}_A = \bar{c}_B = c$ and*

(i) $n_B > \lceil \frac{1}{2} n_A \rceil$, with $\tilde{h}^* = 2n_B$;

(ii) $n_B < \lceil \frac{1}{2} n_A \rceil$, with

$$\begin{aligned}\tilde{h}^* &= n_A \times \frac{2\bar{c}_A n_B(n_A-1) + n_B(n_A-1) + n_A(1-2c)}{2\bar{c}_A[n_A-n_B](n_A-1) + n_B(n_A-1) + n_A(1-2c)} \\ &= n_A \times \xi(c, \bar{c}_A)\end{aligned}$$

where $\xi(c, \bar{c}_A)$ is decreasing in both c and $\bar{c}_A$, and

a) $\xi(c, \bar{c}_A) \in (0, 1)$ for all $0 < c \leq \bar{c}_A < \frac{1}{2}$;

b) $\xi(c, \cdot) \to \frac{2n_B}{n_A}$ as $c \to \frac{1}{2}$, so $\tilde{h}^* \to 2n_B$;

c) $\xi(\cdot, \bar{c}_A) \to 1$ as $\bar{c}_A \to 0$, so $\tilde{h}^* \to n_A$.

Furthermore, $2n_B < \tilde{h}^* < n_A$.

Proof. See A.9. □

Proposition 7 is our second main result. It shows that the maximal expected turnout under correlated equilibria and group-symmetric distributions behaves similarly to the case of a single voting cost, and essentially depends on two things: the relative sizes of the groups and the bounds of the support of the cost distribution. The intuition for the result is similar to Proposition 1. Maximizing turnout implies that constraint (39) for players in N_B binds at the optimum. This in turn implies that constraint (38) for players in N_A binds at the optimum. Now the binding constraint (39) for players in N_B crucially depends on $\bar{c}_B$, because once it holds for the voters with the highest costs in group N_B, it automatically holds for voters in N_B with lower costs. On the other hand, the binding constraint (38) for players in N_A crucially depends on $\underline{c}_A$, because once it holds for the voters with the lowest costs in group N_A, it automatically holds for voters in N_A with higher costs. The effects of the two constraints cancel each other out if and only if $\underline{c}_A = \bar{c}_B$. Once this condition holds, the key difference between cases (i) and (ii) under symmetric distributions only concerns the behavior of constraint (38) for players in N_B and constraint (39) for players in N_A, just like in Proposition 1.

In the proof of Proposition 7 we show that when $\underline{c}_A = \bar{c}_B$, the equilibrium distribution support is the same as in Proposition 1, so Corollary 1 holds without change. For the sake of completeness let us also provide here the expressions for the probability of the largest tie, π_{n_B,n_B}. The only change from Corollary 2 concerns the case of small minority.

Corollary 4. *Suppose $n_A > n_B \geq 1$, $0 < c_i < 0.5$ for all $i \in N$, and $\underline{c}_A = \bar{c}_B = c$. Assuming symmetric distributions,*

(i) if $n_B > \lceil \frac{1}{2} n_A \rceil$ then

$$\pi_{n_B,n_B} = 2c$$

(ii) if $n_B < \lceil \frac{1}{2} n_A \rceil$, then

$$\pi_{n_B,n_B} = \frac{2}{\frac{1}{c}\left[1 + \frac{1}{2\bar{c}_A(n_A-1)} + \frac{n_B}{n_A}\left(\frac{1}{2\bar{c}_A} - 1\right)\right] - \frac{1}{\bar{c}_A(n_A-1)}}$$

Proof. See equations (130) and (151) in the proof of Proposition 7 in A.9. □

Notice that if $c = \bar{c}_A$, the expression for case (ii) coincides with its analogue in Corollary 2.

What happens when $\underline{c}_A \neq \bar{c}_B$? In A.9 we show that if $\bar{c}_B < \underline{c}_A$, then the maximal expected turnout exceeds the value of $\tilde{h}^*$ for both cases of Proposition 7 and for any admissible combination of the other cost thresholds. At first sight this might look counterintuitive: $\bar{c}_B < \underline{c}_A$ implies that the majority group find it costlier to vote than the minority group, so they should vote less. However, the higher voting cost of the majority group also implies that it will be easier to satisfy their IC constraints for abstention, as well as the minority group IC constraints for voting. Thus in the group-symmetric max turnout correlated equilibrium, the competitive profiles with higher total turnout will be assigned higher probabilities, producing higher expected turnout. As $\underline{c}_B \to \frac{1}{2}$, $\bar{c}_B \to \underline{c}_A$, so the maximal expected turnout converges to $\tilde{h}^*$ from above. Similarly, when $\bar{c}_B > \underline{c}_A$, the maximal expected turnout is lower than the value of $\tilde{h}^*$ for both cases of Proposition 7. Nevertheless, as $\min\{\underline{c}_A, \underline{c}_B\} \to \frac{1}{2}$, $\underline{c}_A \to \bar{c}_B$, so the maximal expected turnout converges to $\tilde{h}^*$ from below. Therefore, the result of Proposition 7 is, in a sense, a limiting case when the lowest cost threshold increases towards $\frac{1}{2}$ and symmetric distributions are assumed.

One can also imagine the case where some voters have costs greater than $\frac{1}{2}$ or less than 0. These cases are not very interesting from the analysis point of view: if voter i has a dominant strategy to abstain due to $c_i > \frac{1}{2}$ (violating constraint (39) for any probability distribution that places a positive probability on profiles with i voting), her presence in the list of players does not affect at all the outcome of the election, so we can redefine $N \equiv N \setminus \{i\}$. A more elaborate way to handle this problem requires the use of an asymmetric probability distribution, which would distinguish i from the other players in her group and assign probability zero to all profiles with i voting. We do not fully analyze this case, but we conjecture that allowing for high-cost voters will not substantially change our results.

If voter i has a dominant strategy to vote due to $c_i < 0$, then simply removing this voter results in a loss of generality. The case of negative costs requires some special handling, but it is tractable in our framework. First of all, without additional assumptions about the distribution of such costs across groups, one can nevertheless argue that, under the veil of ignorance, voters with negative costs are just as likely to belong to either of the groups, so we would expect their votes to cancel each other out. Notwithstanding this argument, we would like to consider the case of negative costs for some voters for the following reasons. First, it suggests a turnout model that incorporates some additional factors, like citizen duty, which may be important for some voters. Second, we need to consider the negative costs to be able to directly compare our results with Palfrey and Rosenthal (1985), who in their Assumption 2 explicitly include them. It is important to understand whether we get a high turnout equilibria due to our solution concept being the correlated equilibrium, or due to a different assumption about the cost support.

Let $\mathcal{L} \subset N$ be the set of voters with (strictly) negative costs. We restrict the set of admissible joint distributions to those that place probability zero on voters in $\mathcal{L}$ receiving a recommendation to abstain and probability one on voters in $\mathcal{L}$ receiving a recommendation to vote. With this modification, we can replace the actual group sizes, n_A and n_B with their modified versions, $\tilde{n}_A$ and $\tilde{n}_B$, which take into account the voters from $\mathcal{L}$ so that $\tilde{n}_A = n_A - \mathcal{L}_A$ and $\tilde{n}_B = n_B - \mathcal{L}_B$. This is as if the actual group sizes are shifted by a constant. It is clear that our results hold for the modified game.

4 Incomplete Information

Incomplete information in the voter turnout game was introduced by Ledyard (1981), and further explored in Palfrey and Rosenthal (1985). Under incomplete information, Palfrey and Rosenthal (1985, Theorem 2) established that in the quasi-symmetric Bayesian Nash equilibrium only voters with non-positive voting costs will vote in the limit as n_A, n_B get large. There are several ways to introduce the incomplete information into the basic model, but not all of them are suitable for the analysis of high-turnout correlated equilibria. In this section we consider the simplest version.

In general, player i's *type* is a pair (t_i, c_i) of her political type (candidate preference) and the corresponding cost of voting. The political type directly affects the utilities of all voters through the resulting split into majority and minority, but the voting cost type only affects the utility of a specific player. In this section we assume, for simplicity, that voters' political types are common knowledge.[17] We use t to denote the fixed commonly known joint political type where each voter i has political type t_i. The costs of voting are stochastic: each voter $i \in N$, draws her private cost of voting, c_i, from a commonly known *discrete*[18] distribution F_{t_i} with support $\{\underline{c}_{t_i}, \ldots, \bar{c}_{t_i}\}$, where $0 < \underline{c}_{t_i} \leq \frac{1}{2}$ and $0 < \bar{c}_{t_i} < 1$. The assumption about the support range helps rule out uninteresting equilibria, e.g. those with everyone voting for sure, or those with everyone abstaining for sure. We assume c_i is distributed independently of all other voters' costs c_{-i} (and types t_{-i}). Distributions F_A and F_B determine the set of admissible joint cost profiles, characterized by the tuple of respective cost bounds $(\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B)$ as

$$\mathcal{C}_{(\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B)} \equiv \{(c_i)_{i \in N} | \underline{c}_{t_i} \leq c_i \leq \bar{c}_{t_i}\} \tag{42}$$

[17]This is a strong assumption. There are ways of relaxing it (Myerson, 1998, 2000), but they are inconsistent with the variant of the incomplete information correlated equilibrium we consider in this paper. We conjecture that this assumption can be relaxed in a communication equilibrium (Myerson, 1986; Forges, 1986), and leave it for future research.

[18]Typically it is assumed in the literature that the cost distributions are absolutely continuous. We do not make this assumption to avoid dealing with measurability issues in the definition of a strategic form correlated equilibrium below. See Cotter (1991) for a detailed discussion of these issues.

We write $\mathcal{C}^{-i}_{(\underline{c}_A,\bar{c}_A,\underline{c}_B,\bar{c}_B)}$ to refer to the set of admissible cost profiles for players other than i. Denote $\pi(c)$ the probability of a joint cost profile $c = ((c_i)_{i\in N}) \in \mathcal{C}_{(\underline{c}_A,\bar{c}_A,\underline{c}_B,\bar{c}_B)}$. The independently distributed costs then imply that

$$\pi(c) \equiv \left(\prod_{\{i\in N: t_i=A\}} F_A(c_i)\right)\left(\prod_{\{i\in N: t_i=B\}} F_B(c_i)\right)$$

Since the political types are fixed by assumption, we omit the respective component in the definition of players' strategies and for each $i \in N$ define a pure strategy $s_i : \{\underline{c}_{t_i}, \ldots, \bar{c}_{t_i}\} \to \{0, 1_A, 1_B\}$ as a function that maps voter i's cost into an action (abstain, vote for candidate A, or vote for candidate B, respectively). We assume that voters never vote for the candidate of the opposite political type, so we abuse notation and merge 1_A and 1_B into 1 meaning the act of voting for the "correct" candidate. The set of all pure strategies for player $i \in N$ is a *finite* set $\mathcal{S}_i = \{0,1\}^{\{\underline{c}_{t_i},\ldots,\bar{c}_{t_i}\}}$, i.e. the set of all functions from cost types into actions. Let $\mathcal{S} \equiv \times_{i\in N}\mathcal{S}_i$ be the set of all joint strategies.

The utility of player i from a joint strategy $s(c) \equiv (s_j(c_j)_{j\in N})$ when player i's voting cost is c_i (and the joint political type is t) takes the following form:

$$u_i(s(c)|c_i) = \begin{cases} 1 - s_i(c_i)c_i & \text{if } \sum\limits_{\{j\in N|t_j=t_i\}} s_j(c_j) > \sum\limits_{\{j\in N|t_j\neq t_i\}} s_j(c_j) \\ \frac{1}{2} - s_i(c_i)c_i & \text{if } \sum\limits_{\{j\in N|t_j=t_i\}} s_j(c_j) = \sum\limits_{\{j\in N|t_j\neq t_i\}} s_j(c_j) \\ -s_i(c_i)c_i & \text{if } \sum\limits_{\{j\in N|t_j=t_i\}} s_j(c_j) < \sum\limits_{\{j\in N|t_j\neq t_i\}} s_j(c_j) \end{cases}$$

Let us now discuss the solution concept. There are quite a few alternative definitions of the correlated equilibrium in games with incomplete information (see in particular Forges (1993, 2006, 2009), Section 8.4 of Bergemann and Morris (2013) and Milchtaich (2013)), which are often far from being equivalent. The sets of expected payoffs corresponding to specific definitions are (sometimes) partially ordered by inclusion. We use the strategic form incomplete information correlated equilibrium, as defined in Forges (1993, 2006). This is the strongest definition in the sense that it results in the smallest set of expected payoffs compared, for example, to the communication equilibrium (Myerson (1986), Forges (1986)). Hence if we can obtain a substantial turnout in the strategic form correlated equilibrium, then we can also obtain it in any of the more general definitions of the correlated equilibrium under incomplete information.

A *Strategic Form Incomplete Information Correlated Equilibrium*(SFIICE) is a probability distribution $q \in \Delta(S)$ that selects a pure strategy profile $s = (s_i)_{i \in N}$ with probability $q(s)$, such that when recommended s_i and knowing her type, no player has an incentive to deviate, given that other players follow their recommendations. Formally, $q \in \Delta(S)$ is a SFIICE if for all $i \in N$, all $c_i \in \{\underline{c}_{t_i}, \ldots, \bar{c}_{t_i}\}$, all $a_i \in \{0,1\}$, and any $s_i \in \mathcal{S}_i$ such that $s_i(c_i) = a_i$, we have

$$\sum_{\left\{c_{-i} \in \mathcal{C}^{-i}_{(\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B)}\right\}} \pi(c) \sum_{a_{-i}} \left(\sum_{\{s_{-i}(c_{-i}) = a_{-i}\}} q(s_i, s_{-i}) \right) \left[u_i(a_i, a_{-i}) - u_i(a_i', a_{-i})\right] \geq 0$$

for all $a_i' \in \{0,1\}$.

It will be convenient to explicitly rewrite these conditions as the set of distributions $q \in \Delta(S)$ such that for all $i \in N$, all $c_i \in \{\underline{c}_{t_i}, \ldots, \bar{c}_{t_i}\}$, and all $s_i \in \mathcal{S}_i$ such that $s_i(c_i) = 0$ we have

$$\begin{aligned} \sum_{c_{-i}} \pi(c) \Bigg[c_i \sum_{a_{-i} \in V_D^i} & \left(\sum_{\{s_{-i}(c_{-i}) = a_{-i}\}} q(s_i, s_{-i} | s_i(c_i) = 0) \right) \\ & + \left(c_i - \frac{1}{2}\right) \sum_{a_{-i} \in V_P^i} \left(\sum_{\{s_{-i}(c_{-i}) = a_{-i}\}} q(s_i, s_{-i} | s_i(c_i) = 0) \right) \Bigg] \geq 0 \end{aligned} \tag{43}$$

and for all $s_i \in S_i$ such that $s_i(c_i) = 1$ we have

$$\begin{aligned} \sum_{c_{-i}} \pi(c) \Bigg[-c_i \sum_{a_{-i} \in V_D^i} & \left(\sum_{\{s_{-i}(c_{-i}) = a_{-i}\}} q(s_i, s_{-i} | s_i(c_i) = 1) \right) \\ & + \left(\frac{1}{2} - c_i\right) \sum_{a_{-i} \in V_P^i} \left(\sum_{\{s_{-i}(c_{-i}) = a_{-i}\}} q(s_i, s_{-i} | s_i(c_i) = 1) \right) \Bigg] \geq 0 \end{aligned} \tag{44}$$

where, as before, V_P^i and V_D^i are the set of joint action profiles such that player i is pivotal and dummy, respectively, and the summation over the others' costs is understood to be over cost profiles in $\mathcal{C}^{-i}_{(\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B)}$. The induced probability distribution over action profiles at every cost profile $c \in \mathcal{C}_{(\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B)}$ is given by

$$\nu(a|c) \equiv \sum_{\{s \in S | \forall i \in N : s_i(c_i) = a_i\}} q(s) \tag{45}$$

The max turnout problem under incomplete information now takes the following form:

$$g^* \equiv \max_{q \in \mathcal{D}(N_A, N_B, F_A, F_B)} \sum_{\{c \in \mathcal{C}_{(\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B)}\}} \pi(c) \left(\sum_{a \in \{0,1\}^n} \nu(a|c) \left(\sum_{i \in N} a_i \right) \right) \tag{46}$$

Full characterization of the solution to this problem is not our goal in this section. Rather, we just want to show a possibility result, that correlated equilibria with substantial turnout can survive in the incomplete information case. The next proposition delivers the desired result.

Proposition 8. *Suppose* $n_A, n_B \geq 1$ *and* $n_A > n_B$*. Let* F_A, F_B *be any discrete distributions over players' voting costs,* $\{\underline{c}_A, \ldots, \bar{c}_A\}$*, and* $\{\underline{c}_B, \ldots, \bar{c}_B\}$*, respectively, such that* $\bar{c}_B \leq \underline{c}_A \in (0, 0.5)$*,* $0 < \bar{c}_A < 0.5$*, and* $0 < \underline{c}_B < 0.5$*. Then* $g^* \geq \tilde{h}^*$*, where* $\tilde{h}^*$ *is defined in* (41)*.*

Proof. See A.10. □

This result holds for large electorates as well.[19]

5 Discussion

Since Nash equilibria are also correlated equilibria, it is important to understand what exactly the analysis of correlated equilibria adds to the existing results in the literature.

Under complete information and common voting cost, our paper extends Palfrey and Rosenthal (1983), who characterized two classes of the Nash equilibria that exhibit substantial turnout and survive when the electorate becomes large.[20] Palfrey and Rosenthal call those *mixed-pure strategy equilibria* and *symmetric totally-mixed strategy equilibria*, respectively. The former equilibria require all voters in one group mixing between voting and abstention with some common probability, whereas voters in the other group are further divided into two subgroups such that all voters in one subgroup vote for sure, and all voters in the other subgroup abstain for sure. The latter equilibria require voters in each group mixing with the same group-specific probability. Both of these equilibrium classes have a counter-intuitive property: the expected turnout is *increasing* in cost. Furthermore, symmetric totally-mixed equilibria only exist when the cost is large enough

[19] In particular, we mean the case where the ratio between group sizes remains fixed as their sizes increase to ∞, with fixed cost supports.

[20] This latter criterion is important: Palfrey and Rosenthal (1983) identify many other equilibria that have nice properties, but do not survive in large electorates.

and both groups have the same size. This unfortunate dependence on both groups having exactly the same size translates directly into the incomplete information case, and, in a sense, is the primary reason why no high-turnout equilibria survive even slightest uncertainty in Palfrey and Rosenthal (1985) when the electorate size gets large. The corresponding result in this paper (see Proposition 1) has neither of these shortcomings.

Under heterogeneous voting costs, we can compare our Proposition 7 with Taylor and Yildirim (2010, Proposition 2). They find that under incomplete information, in large electorates the limit expected turnout and the probability of winning are completely determined by the lowest voting costs in each group. In contrast, the max turnout correlated equilibrium puts a joint restriction both on the lowest voting cost in one group and the highest voting cost in the other. This is the effect of two opposing incentive compatibility constraints. In a quasi-symmetric Bayesian Nash equilibrium in cutpoint strategies, which is typically considered in the literature, the two constraints for each group merge into one at the critical cost. Another related result is Kalandrakis (2007), who proves that under complete information and heterogeneous costs almost all Nash equilibria are regular, and there exists at least one monotone Nash equilibrium, where players with higher costs participate with weakly lower probabilities. In our group-symmetric max-turnout correlated equilibria a similar logic allows to restrict attention to the lowest and highest costs in each group.

Under incomplete information, we extend Palfrey and Rosenthal (1985). Their high-turnout equilibria do not survive uncertainty when the electorate size gets large. In contrast, our high-turnout correlated equilibria persist under certain conditions on cost supports (see Proposition 8). This result can be also compared with Kalandrakis (2009), who basically shows that high turnout Nash equilibria of the complete information game with a common positive cost can persist under incomplete information. Assuming that densities of the private voting cost, private benefit, or both, converge to a point mass that corresponds to a complete information turnout game with a positive common voting cost, Kalandrakis (2009, Thm 4) permits introducing incomplete information with respect to individual voting cost, the size of each candidate's support, or both. The crucial difference from Palfrey and Rosenthal (1985)'s negative result on high-turnout equilibria under incomplete information is that Kalandrakis holds the size of the electorate fixed, and varies the uncertainty level, while Palfrey and Rosenthal hold the uncertainty level fixed and vary the total size of the electorate. A natural restriction on Kalandrakis' results comes from the fact that the Nash equilibria of the complete information game can be approximated by Bayesian equilibria only for sufficiently small perturbations. Thus while Kalandrakis (2009) established regularity of the class of asymmetric Nash equilibria which was typically dismissed in the literature due to lack of tractability, he does not resolve the turnout paradox. Our results, in a sense, provide a link between those two papers. We show in Proposition 8 that group-symmetric max-turnout correlated equilibria can be preserved under incomplete information about voting costs, while correlation allows to maintain high-turnout for large electorates, as long as cost supports are fixed.

One potential criticism of our model concerns the idea of maximizing the expected total turnout without separate considerations for turnout in each group of supporters. It is not clear a priori whether the our model can be consistent with the models of the group-based ethical voter approach, if we assume that both groups independently maximize the turnout among their own members. However, our results in Proposition 1 show that when the minority is large, the same level of maximal expected total turnout can be achieved when groups maximize their members' turnout independently. We relegate more general analysis to a companion paper (Pogorelskiy, 2015), which explicitly addresses coordination among groups in a new equilibrium concept.

6 Concluding remarks

This is the first paper to introduce and characterize the set of correlated equilibria in the voter turnout games. The solution concept of the correlated equilibrium, developed by Aumann (1974, 1987), allows us to explicitly take into account the possibilities of pre-play communication between voters. Communication expands the set of equilibrium outcomes in turnout games thereby providing a micro foundation for group-based mobilization, as well as a solution to the turnout paradox that does not require ad hoc assumptions about voters' utility.

We analyzed the correlated equilibrium turnout in three main settings, varying the information structure (complete and incomplete) and the assumptions on agents' voting costs (homogenous and heterogeneous).

Under complete information and homogenous voting cost, we fully characterized the turnout bounds in terms of the correlated equilibria that maximize and minimize the expected turnout. These bounds provide a theoretical constraint on the levels of turnout that can be achieved if there are no restrictions on pre-play communication, and also characterize the range of expected turnout implementable in a correlated equilibrium. The set of correlated equilibria includes all equilibria arising under any of the more restricted communication protocols, e.g., voter communication in networks.

We found that there are two classes of the max turnout correlated equilibria, determined by the relative sizes of the two competing groups. If the minority is at least half the size of the majority, the resulting expected turnout is twice the size of the minority and does not depend on the cost. If the minority is less than half the size of the majority, the resulting expected turnout is a decreasing function of the voting cost that starts at the size of the majority for low costs and goes down to twice the size of the minority for high costs. We also characterized the equilibrium distribution support and several key election statistics (probabilities of a tie and of the majority winning). In contrast to the high-turnout Nash equilibria, the high-turnout correlated equilibria possess intuitive properties. For example, the majority group is more likely to win for all costs, and the tie probability is increasing in the cost. We also characterize the correlated equilibria that maximize social welfare. Those are generally different from the minimal turnout equilibria, but exhibit a similar range of expected turnout.

We then showed that the high-turnout equilibria under complete information and homogenous voting cost have analogues under heterogenous costs, which may also remain feasible correlated equilibria under incomplete information about voting costs (assuming certain additional conditions about the cost support).

Our results emphasize the important role of communication in turnout games. A natural question remains: why is the correlated equilibrium a reasonable solution concept? How, exactly, the correlated equilibria we describe in this paper can be implemented? The answer to the first question is given by Aumann (1987) and Hart and Mas-Colell (2000). Correlated equilibrium can be interpreted as an "expression of Bayesian rationality": if it is common knowledge that every player maximizes expected utility given her (subjective) beliefs about the state of the world, the resulting strategy choices form a correlated equilibrium. Furthermore, correlated equilibrium can be obtained as a result of a simple dynamic procedure driven by players' regret over past period observations.

The answer to the second question typically invokes describing a direct mechanism where an impartial mediator, such as a leader, gives recommendations to players. However it is important to realize that a correlated equilibrium can be also implemented without the mediator, as a Nash (or even sequential) equilibrium of the expanded game with simple communication.[21] Laboratory experiments are a useful tool for understanding the effects of unmediated communication on turnout in a controlled setting. Elsewhere (Palfrey and Pogorelskiy, 2014) we show that these effects are nuanced: with a low voting cost, party-restricted communication increases turnout, while public communication decreases turnout; while with a high voting cost, public communication increases turnout. From a theoretical perspective, establishing a realistic communication scheme

[21] See Forges (1990), Gerardi (2004), and Gerardi and Myerson (2007).

that is "minimally necessary" for implementing high-turnout correlated equilibria remains a promising extension that we leave for future research.

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Appendix

Appendix A Proofs

A.1 Proof of Lemma 1

Proof. Fix any $i \in N_A$ and consider two voting profiles: $x_1 := (0_i, a, b)$ and $x_2 := (1_i, a-1, b)$ such that the total number of votes in group N_A is a, the total number of votes in group N_B is b, and in the first profile voter i abstains, while in the second profile i turns out to vote and somebody else from N_A abstains. We will construct the equivalent symmetric distribution iteratively. At step 1, we let $\sigma_1^*(s) = \mu^*(s)$ for all profiles $s \neq x_1, x_2$. The objective in either (9) or (10) does not depend on voters' identities, only on the total number of votes in each profile. Since the total number of votes at either x_1 or x_2 is the same and equals $a+b$, it does not matter for the objective how σ_1^* distributes the probability mass among x_1 and x_2 compared to μ^* as long as $\mu^*(x_1)+\mu^*(x_2) = \sigma_1^*(x_1)+\sigma_1^*(x_2)$. Hence we can let $\sigma_1^*(x_1) = \sigma_1^*(x_2) = \frac{1}{2}(\mu^*(x_1)+\mu^*(x_2))$. Clearly, this argument holds for any $a \in \{1, \ldots, n_A - 1\}$, any $b \in \{0, \ldots, n_B\}$, and any $i \in N_A$, and a similar argument holds for any $k \in N_B$ and profiles $(0_k, a, b)$ and $(1_k, a, b-1)$, respectively. We can now iteratively construct σ^*, where at each step $t \geq 2$ we define x_1^t and x_2^t by one of the remaining combinations of (a, b, i), and let $\sigma_t^*(s) = \sigma_{t-1}^*(s)$ for all profiles $s \neq x_1^t, x_2^t$. Once we have considered all combinations, we obtain σ^*, for which by construction $f(\sigma^*) = f(\mu^*)$ and $\sigma^* \in \mathcal{M}$. It remains to show that all IC constraints are satisfied at σ^*. To see this, let's roll back to σ_1^* and show that the IC constraints are satisfied at each iteration. Notice that the sets V_D^i and V_P^i in (8) and (7) do not depend on other voters' identities, but only on the total number of votes on each side of the profile, hence $x_1 \in V_P^i$ if and only if $x_2 \in V_P^\ell$ for any $\ell \in N_A, \ell \neq i$ such that ℓ votes at x_1 and abstains at x_2 (since $a \in \{1, \ldots, n_A - 1\}$, there must exist at least one such player). Similarly, $x_2 \in V_P^i$ if and only if $x_1 \in V_P^\ell$ for any such ℓ. These relations hold for all (x_1, x_2) with $a \in \{1, \ldots, n_A - 1\}$, and any $b \in \{0, \ldots, n_B\}$. By assumption, IC constraints (5)-(6) hold for all $i \in N$ under μ^*. Since the voting cost is the same for everyone in N_A, and μ^* is optimal, the corresponding IC constraints must be of the same kind (slack or binding) for both i and ℓ under μ^*, and, moreover, they must put the same restriction on the total probability that i is pivotal at x_1 as they put on the total probability that ℓ is pivotal at x_2, i.e.

$$\sum_{\substack{s_{-i} \in V_P^i \\ |s_{-i}| = a+b}} \mu^*(0_i, 1_\ell, s_{-i\cup\ell}) = \sum_{\substack{s_{-\ell} \in V_P^\ell \\ |s_{-\ell}| = a+b}} \mu^*(0_\ell, 1_i, s_{-i\cup\ell})$$

and

$$\sum_{\substack{s_{-i}\in V_P^i\\ |s_{-i}|=a-1+b}} \mu^*(1_i, 0_\ell, s_{-i\cup\ell}) = \sum_{\substack{s_{-\ell}\in V_P^\ell\\ |s_{-\ell}|=a-1+b}} \mu^*(1_\ell, 0_i, s_{-i\cup\ell})$$

But then redistributing this probability mass symmetrically under σ^* does not violate the IC for $i \in N_A$, $a \in \{1, \ldots, n_A - 1\}$, and any $b \in \{0, \ldots, n_B\}$. Similarly, we can prove that this redistribution does not violate the IC for $k \in N_B$ and $b \in \{1, \ldots, n_B - 1\}, a \in \{0, \ldots, n_A\}$. □

A.2 Proof of Proposition 1

Proof. Using the fact that all profile probabilities sum up to one and $\mu_{0,0} = 0$ at the optimum, rewrite the objective in (34) as

$$\begin{aligned}1 + \sum_{\{s|\sum s_i=2\}} \mu(s) + 2 \sum_{\{s|\sum s_i=3\}} \mu(s) + \ldots \\ +(n-2) \sum_{\{s|\sum s_i=n-1\}} \mu(s) + (n-1)\mu(1,\ldots,1)\end{aligned} \tag{47}$$

Since $\sum_s \mu(s) = 1$, the above expression is maximized if the largest possible probability is placed on the outcomes with more turnout.[22] In particular, the maximal possible value of n is achieved when $\mu(1, \ldots, 1) = 1$.

Since $n_A > n_B$, the full turnout profile, $(1, \ldots, 1)$ is in V_A. By Lemma 1, it is sufficient to consider symmetric distributions. To simplify the notation, we denote the probability of any profile with a, b total votes for A, B, respectively, by $\mu_{a,b} \equiv \mu(\#A = a, \#B = b)$, without further reference to an individual player. We are going to use these $(n_A + 1)(n_B + 1)$ probabilities as our decision variables. When we distinguish between individual voters among those in the profile (a, b), however, there are going to be $\binom{n_A}{a}\binom{n_B}{b}$ different profiles (each having the same probability $\mu_{a,b}$ in the symmetric distribution). Hence the total probability constraint is now written as

$$\sum_{a=0}^{n_A}\sum_{b=0}^{n_B} \binom{n_A}{a}\binom{n_B}{b}\mu_{a,b} = 1 \tag{48}$$

One may wonder how the symmetric distribution can be implemented. In the mediator setup, we can think of it in the following way: a mediator picks a voting profile (a, b) with probability $\binom{n_A}{a}\binom{n_B}{b}\mu_{a,b}$, and then randomly recruits the respective number

[22] Indeed, each consecutive term in the expanded sum has a greater marginal effect on the value of the objective than the previous term.

of voters on each side. These voters receive a recommendation to vote. The remaining voters receive a recommendation to abstain.

Using the symmetry, we can rewrite constraints (5)-(6) for players in N_A (N_B, respectively) as the following system of four inequalities with respect to $(n_A+1)(n_B+1)$ variables of the form $\mu_{a,b}$:

$$\sum_{a=1}^{n_A-1}\sum_{b=0}^{\min\{a-1,n_B\}}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b}+\sum_{a=0}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b}\geq \frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a,a}+\sum_{a=0}^{n_B-1}\binom{n_A-1}{a}\binom{n_B}{a+1}\mu_{a,a+1}\right) \tag{49}$$

$$\sum_{a=2}^{n_A}\sum_{b=0}^{\min\{a-2,n_B\}}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b}+\sum_{a=1}^{n_B-1}\sum_{b=a+1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b}\leq \frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a}+\sum_{a=1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{a}\mu_{a,a}\right) \tag{50}$$

and

$$\sum_{a=2}^{n_A}\sum_{b=0}^{\min\{a-2,n_B-1\}}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b}+\sum_{a=0}^{n_B-2}\sum_{b=a+1}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b}\geq \frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{a}\mu_{a,a}+\sum_{a=0}^{n_B-1}\binom{n_A}{a+1}\binom{n_B-1}{a}\mu_{a+1,a}\right) \tag{51}$$

$$\sum_{a=2}^{n_A}\sum_{b=1}^{\min\{a-1,n_B\}}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}+\sum_{a=0}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}\leq \frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{a}\mu_{a,a+1}+\sum_{a=1}^{n_B}\binom{n_A}{a}\binom{n_B-1}{a-1}\mu_{a,a}\right) \tag{52}$$

We will refer to the first and the third inequality above as the odd incentive compatibility (IC) constraints, and to the second and the fourth inequality as the even IC constraints, distinguished by the group.

Since we assumed $n_A > n_B$, at the largest turnout profile $\mu(1,\ldots,1)\equiv\mu_{n_A,n_B}$ voters from N_B (as well as voters from N_A, if $n_A > n_B+1$) are dummies. This implies that the even IC constraint for N_B is always binding at the optimum. As for the even IC constraint for N_A, we can show that for $n_A > n_B \geq \lceil \frac{1}{2}n_A\rceil$ it is always slack. To see this, notice that the even IC for N_A requires

$$\begin{aligned}
\mu_{n_A,n_B} \leq \frac{\frac{1}{2}-c}{c} \Bigg(& \sum_{a=0}^{n_B-1} \binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a} + \sum_{a=1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{a}\mu_{a,a}\Bigg) \\
& - \Bigg[\sum_{b=1}^{n_B-1}\binom{n_B}{b}\mu_{n_A,b} + \sum_{a=3}^{n_B}\sum_{b=1}^{a-2}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b} \\
& + \sum_{a=n_B+2}^{n_A-1}\sum_{b=1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b} + \sum_{a=1}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b}\Bigg] \\
& - \sum_{a=1}^{n_B-1}\binom{n_A-1}{a-1}\binom{n_B}{a+1}\mu_{a,a+1} - \sum_{b=1}^{n_B}\binom{n_A-1}{n_B}\binom{n_B}{b}\mu_{n_B+1,b} - \sum_{a=2}^{n_B}\binom{n_A-1}{a-1}\mu_{a,0} \\
& + \frac{1}{2c}\binom{n_A-1}{n_B}\mu_{n_B+1,n_B} - \sum_{a=n_B+1}^{n_A}\binom{n_A-1}{a-1}\mu_{a,0}
\end{aligned} \tag{53}$$

The binding even IC for N_B requires

$$\begin{aligned}
\mu_{n_A,n_B} = \frac{\frac{1}{2}-c}{c} \Bigg(& \sum_{a=0}^{n_B-1} \binom{n_A}{a}\binom{n_B-1}{a}\mu_{a,a+1} + \sum_{a=1}^{n_B}\binom{n_A}{a}\binom{n_B-1}{a-1}\mu_{a,a}\Bigg) \\
& - \Bigg[\sum_{b=1}^{n_B-1}\binom{n_B-1}{b-1}\mu_{n_A,b} + \sum_{a=3}^{n_B}\sum_{b=1}^{a-2}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b} \\
& + \sum_{a=n_B+2}^{n_A-1}\sum_{b=1}^{n_B}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b} + \sum_{a=1}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}\Bigg] \\
& - \sum_{a=1}^{n_B-1}\binom{n_A}{a+1}\binom{n_B-1}{a-1}\mu_{a+1,a} - \sum_{b=1}^{n_B}\binom{n_A}{n_B+1}\binom{n_B-1}{b-1}\mu_{n_B+1,b} \\
& - \sum_{b=2}^{n_B}\binom{n_B-1}{b-1}\mu_{0,b}
\end{aligned} \tag{54}$$

Comparing the right hand sides of these two expressions, we see that in every single term of (53), except for the two terms on the last line, the total profile turnout matches exactly the total profile turnout in the corresponding term of (54). There are three possibilities. If the RHS of (54) is strictly less than the RHS of (53), the even IC for N_A is slack, so we are done. The RHS of (54) cannot be strictly greater, since then the even IC for N_A does not hold at all, and so we are not at the optimum of the constrained maximization program. The critical case is when the two RHS are the same; but this holds at the optimum if and only if the sum of the last two terms of (53) is zero (otherwise, since the profiles in the last two terms of (53) are not matched in (54), and so are not restricted by (54), the RHS of (53) in the optimum can be increased without increasing the RHS of (54), which is optimal when $n_B \geq \lceil \frac{1}{2}n_A \rceil$).

As long as $n_B \geq \lceil \frac{1}{2} n_A \rceil$, we have $2n_B \geq n_A$ and $2n_B + 1 > n_A$, so

$$\frac{1}{2c}\binom{n_A - 1}{n_B}\mu_{n_B+1,n_B} - \sum_{a=n_B+1}^{n_A}\binom{n_A-1}{a-1}\mu_{a,0} > 0$$

That is, the sum of the two terms on the last line of (53) is strictly positive at the optimum. This follows, since the total turnout of the first term, $2n_B + 1$, exceeds the total turnout of the largest term in the above sum, which is n_A, achieved at $\mu_{n_A,0}$. Therefore, the RHS of (54) is strictly less at the optimum than the RHS of (53) and so the even IC for N_A is slack, given that the even IC for N_B is binding. Now, if $n_B < \lceil \frac{1}{2} n_A \rceil$, then $2n_B + 1 \leq n_A$, so it is easy to see that the even IC constraints for both groups are binding at the optimum.

As for the odd IC constraints, we can show that the situation is the opposite: the odd IC for N_A is always binding at the optimum, while the odd IC for N_B only binds when $n_B > \lceil \frac{1}{2} n_A \rceil$ (for even n_A) or $n_B \geq \lceil \frac{1}{2} n_A \rceil$ (for odd n_A). To see this, notice that in the binding constraint (54) all profiles such that a non-voter from N_A is a dummy have the negative sign, so we want to reduce them as much as possible in the optimum. The only subset of profiles where a non-voter from N_A is a dummy which is not directly restricted by (54) has the form $\sum_{a=1}^{n_A-1}\binom{n_A-1}{a-1}\mu_{a,0}$. But these profiles are restricted by (53). If the latter is binding, the restriction is trivial. Suppose not, then if we reduced all directly restricted by (54) probabilities to their lower limit of zero and the odd IC for N_A still was not binding, then constraint (53) (slack by assumption) would imply that $\mu_{n_A,n_B} < 0$. Therefore, the odd IC for N_A must bind at the optimum.

Let us now turn to the odd IC constraint for N_B. The odd IC for N_A is binding as we just demonstrated, so we can rewrite (49) and (51), respectively, as

$$\begin{aligned}
\sum_{a=n_B+1}^{n_A-1}\binom{n_A-1}{a}\mu_{a,n_B} - \frac{\frac{1}{2}-c}{c}\binom{n_A-1}{n_B}\mu_{n_B,n_B} \\
+\binom{n_A-1}{n_B}\binom{n_B}{n_B-1}\mu_{n_B,n_B-1} + \sum_{a=0}^{n_B-2}\binom{n_A-1}{a}\mu_{a,n_B} \\
+\sum_{a=0}^{n_B-2}\binom{n_A-1}{a+1}\binom{n_B}{a}\mu_{a+1,a} + \sum_{a=2}^{n_B+1}\sum_{b=0}^{a-2}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b} \\
+\sum_{a=n_B+2}^{n_A-1}\sum_{b=0}^{n_B-1}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b} + \sum_{a=0}^{n_B-3}\sum_{b=a+2}^{n_B-1}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b} = \\
\frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B-1}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a,a} + \sum_{a=0}^{n_B-1}\binom{n_A-1}{a}\binom{n_B}{a+1}\mu_{a,a+1}\right) \qquad (55)
\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{b=0}^{n_B-1}\binom{n_B-1}{b}\mu_{n_A,b}\\
&+\sum_{a=0}^{n_B-2}\binom{n_A}{a}\binom{n_B-1}{a+1}\mu_{a,a+1}+\sum_{a=2}^{n_B+1}\sum_{b=0}^{a-2}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b}\\
&+\sum_{a=n_B+2}^{n_A-1}\sum_{b=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b}+\sum_{a=0}^{n_B-3}\sum_{b=a+2}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b}\geq\\
&\frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{a}\mu_{a,a}+\sum_{a=0}^{n_B-1}\binom{n_A}{a+1}\binom{n_B-1}{a}\mu_{a+1,a}\right)
\end{aligned}
\tag{56}
$$

Comparing these two expressions, we see that, except for the terms on the first two lines of (55) and those on the first line of (56), in every remaining profile of (55) the total turnout matches exactly the total turnout in the corresponding term of (56).

Suppose $n_B < \lceil \frac{1}{2} n_A \rceil$, then $2n_B < n_A$. We want to show that at the optimum

$$
\begin{aligned}
&\sum_{b=0}^{n_B-1}\binom{n_B-1}{b}\mu_{n_A,b}+\frac{1}{2c}\binom{n_A-1}{n_B}\mu_{n_B,n_B}>\\
&\sum_{a=n_B+1}^{n_A-1}\binom{n_A-1}{a}\mu_{a,n_B}+\binom{n_A-1}{n_B}\mu_{n_B,n_B}\\
&+\binom{n_A-1}{n_B}\binom{n_B}{n_B-1}\mu_{n_B,n_B-1}+\sum_{a=0}^{n_B-2}\binom{n_A-1}{a}\mu_{a,n_B}
\end{aligned}
\tag{57}
$$

The case of $n_A - 1 < n_B + 1$ is not possible, since then $n_A = n_B + 1$, but $2n_B < n_A$ implies $n_B < 1$. So $n_A - 1 \geq n_B + 1$, then $n_A \geq n_B + 2$. Notice that on the RHS of (57) (at profiles with probability μ_{a,n_B} in the first sum) the total turnout in each profile equals $n_B + n_A - k$, where $k \geq 1$, matching the corresponding turnout in each profile on the LHS of (57) (at profiles with probability $\mu_{n_A,b}$) as long as $n_A - k \geq n_B + 1$ (since $a \geq n_B + 1$ in the first sum). Once $n_A - k = n_B + 1$, there are no more profiles left in the first sum of the RHS of (57), but there remain profiles with probability $\mu_{n_A,b}$ in the corresponding sum on the LHS of (57) as long as $0 \leq n_B - k \leq n_B - 1$, since we have $0 \leq b \leq n_B - 1$. Writing the largest possible $k^* = n_A - n_B - 1$, we see that since $n_A \geq n_B + 2$ by assumption, we indeed have $k^* \geq 1$. Therefore, the LHS of (57) contains the profiles with larger turnout that are unmatched by the profiles on the RHS of (57): at the very least, the corresponding probabilities are $\mu_{n_A,0}$ and $\mu_{n_A,1}$. So at the optimum (57) holds; hence, the odd IC for N_B is slack.

Now if n_A is even, we can extend this result to the case where $n_B = \lceil \frac{1}{2} n_A \rceil$, since then $2n_B = n_A$, so even though $\mu_{n_A,1}$ becomes matched by the first probability in the sum on the RHS, μ_{n_B+1,n_B}, we still have $\mu_{n_A,0}$ unmatched on the LHS. However, if n_A is odd, then $n_B = \lceil \frac{1}{2} n_A \rceil$ implies $2n_B = n_A + 1$, so $\mu_{n_A,0}$ becomes matched by μ_{n_B,n_B-1}, and hence the odd IC for N_B is binding at the optimum.

Finally, if $n_B > \lceil \frac{1}{2} n_A \rceil$, then $2n_B \geq n_A + 1$, so all profiles on the LHS of (57) are matched by the corresponding profiles on the RHS, so the odd IC for N_B is binding.

Table 1 summarizes our findings on binding and slack constraints in the maximization problem. To finish the proof, we need to consider three cases, corresponding to the

Table 1: IC constraints at the optimum (max-turnout equilibria)

	$n_B < \lceil \frac{1}{2} n_A \rceil$	$n_B = \lceil \frac{1}{2} n_A \rceil$	$n_B > \lceil \frac{1}{2} n_A \rceil$
Odd IC for N_A (49)	always binds		
Even IC for N_A (50)	binds	slack	slack
Odd IC for N_B (51)	slack	slack for even n_A; binds for odd n_A	binds
Even IC for N_B (52)	always binds		

Note: $n_A > n_B$, $0 < c < \frac{1}{2}$

columns of Table 1.

First, suppose $n_B > \lceil \frac{1}{2} n_A \rceil$. Then the odd IC constraint for N_A binding implies

$$\sum_{a=1}^{n_B+1} \sum_{b=0}^{a-1} \binom{n_A-1}{a} \binom{n_B}{b} \mu_{a,b} + \sum_{a=n_B+2}^{n_A-1} \sum_{b=0}^{n_B} \binom{n_A-1}{a} \binom{n_B}{b} \mu_{a,b}$$
$$+ \sum_{a=0}^{n_B-2} \sum_{b=a+2}^{n_B} \binom{n_A-1}{a} \binom{n_B}{b} \mu_{a,b} - \frac{\frac{1}{2}-c}{c} \left(\sum_{a=0}^{n_B} \binom{n_A-1}{a} \binom{n_B}{a} \mu_{a,a} \right.$$
$$\left. + \sum_{a=0}^{n_B-1} \binom{n_A-1}{a} \binom{n_B}{a+1} \mu_{a,a+1} \right) = 0, \tag{58}$$

the odd IC constraint for N_B binding implies

$$\sum_{a=2}^{n_B+1} \sum_{b=0}^{a-2} \binom{n_A}{a} \binom{n_B-1}{b} \mu_{a,b} + \sum_{a=n_B+2}^{n_A} \sum_{b=0}^{n_B-1} \binom{n_A}{a} \binom{n_B-1}{b} \mu_{a,b}$$
$$+ \sum_{a=0}^{n_B-2} \sum_{b=a+1}^{n_B-1} \binom{n_A}{a} \binom{n_B-1}{b} \mu_{a,b} - \frac{\frac{1}{2}-c}{c} \left(\sum_{a=0}^{n_B-1} \binom{n_A}{a} \binom{n_B-1}{a} \mu_{a,a} \right.$$
$$\left. + \sum_{a=0}^{n_B-1} \binom{n_A}{a+1} \binom{n_B-1}{a} \mu_{a+1,a} \right) = 0, \tag{59}$$

and the even IC constraint for N_B binding implies

$$\begin{aligned}\mu_{n_A,n_B} = \frac{\frac{1}{2}-c}{c}&\left(\sum_{a=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{a}\mu_{a,a+1}+\sum_{a=1}^{n_B}\binom{n_A}{a}\binom{n_B-1}{a-1}\mu_{a,a}\right)\\ &-\left[\sum_{b=1}^{n_B-1}\binom{n_B-1}{b-1}\mu_{n_A,b}+\sum_{a=2}^{n_B}\sum_{b=1}^{a-1}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}\right.\\ &\left.+\sum_{a=n_B+1}^{n_A-1}\sum_{b=1}^{n_B}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}+\sum_{a=0}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}\right] \qquad (60)\end{aligned}$$

At the optimum, μ_{n_A,n_B} must be as large as possible. This implies that the terms in the first parentheses must be as large as possible, and in particular, the last term in the second sum, $\binom{n_A}{n_B}\mu_{n_B,n_B}$, since it has the largest turnout among the terms with the positive sign. Now, from the odd IC for N_A,

$$\begin{aligned}\binom{n_A-1}{n_B-1}\cdot\mu_{n_B-1,n_B} = \frac{c}{\frac{1}{2}-c}&\left[\sum_{a=1}^{n_B+1}\sum_{b=0}^{a-1}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b}\right.\\ &\left.+\sum_{a=n_B+2}^{n_A-1}\sum_{b=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b}+\sum_{a=0}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b}\right]\\ &-\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a,a}+\sum_{a=0}^{n_B-2}\binom{n_A-1}{a}\binom{n_B}{a+1}\mu_{a,a+1}\right) \qquad (61)\end{aligned}$$

Substituting, re-arranging and simplifying the terms (notice that $\mu_{0,0}=0$ by optimality),

$$\begin{aligned}\mu_{n_A,n_B} = \frac{\frac{1}{2}-c}{c}&\left(-\sum_{a=0}^{n_B-2}\binom{n_A}{a}\binom{n_B-1}{a+1}\frac{n_A+1}{n_A-n_B+1}\mu_{a,a+1}\right.\\ &\left.+\sum_{a=1}^{n_B}\binom{n_A}{a}\binom{n_B}{a}\mu_{a,a}\left(\frac{a(n_A+1)-n_An_B}{n_B(n_A-n_B+1)}\right)\right)\\ &+\sum_{a=0}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{(n_B-b)(n_A+1)+(b-a-1)n_B}{n_B(n_A-n_B+1)}\right)\mu_{a,b}\\ &+\sum_{a=n_B+2}^{n_A-1}\sum_{b=1}^{n_B}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{n_B(n_A+b-a)-b(n_A+1)}{n_B(n_A-n_B+1)}\right)\mu_{a,b}\\ &+\sum_{a=2}^{n_B}\sum_{b=1}^{a-1}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{n_B(n_A+b-a)-b(n_A+1)}{n_B(n_A-n_B+1)}\right)\mu_{a,b}\\ &+\sum_{b=1}^{n_B}\binom{n_A}{n_B+1}\binom{n_B}{b}\left(\frac{(n_A-n_B)(n_B-b)-(n_B+b)}{n_B(n_A-n_B+1)}\right)\mu_{n_B+1,b}\\ &+\frac{n_A}{n_A-n_B+1}\sum_{a=1}^{n_A-1}\binom{n_A-1}{a}\mu_{a,0}-\sum_{b=1}^{n_B-1}\binom{n_B-1}{b-1}\mu_{n_A,b} \qquad (62)\end{aligned}$$

The binding odd IC for N_B allows us to express μ_{n_A,n_B-1} as

$$\mu_{n_A,n_B-1} = \frac{\frac{1}{2}-c}{c}\left(\sum_{a=1}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{a}\mu_{a,a} + \sum_{a=0}^{n_B-1}\binom{n_A}{a+1}\binom{n_B-1}{a}\mu_{a+1,a}\right)$$
$$- \left(\sum_{a=2}^{n_B+1}\sum_{b=0}^{a-2}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b} + \sum_{b=0}^{n_B-2}\binom{n_B-1}{b}\mu_{n_A,b}\right.$$
$$\left. + \sum_{a=n_B+2}^{n_A-1}\sum_{b=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b} + \sum_{a=0}^{n_B-2}\sum_{b=a+1}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b}\right)$$

Plugging in μ_{n_A,n_B-1} into the expression for μ_{n_A,n_B} above, we obtain

$$\mu_{n_A,n_B} = \frac{\frac{1}{2}-c}{c}\frac{1}{n_A-n_B+1}\binom{n_A}{n_B}\mu_{n_B,n_B} + \sum_{a=n_B+1}^{n_A-1}\binom{n_A}{a}\left(\frac{n_B-a-1}{n_A-n_B+1}\right)\mu_{a,n_B}$$
$$+\frac{\frac{1}{2}-c}{c}\sum_{a=1}^{n_B-1}\binom{n_A}{a}\binom{n_B}{a}\left(\frac{(a-n_B)(2n_A-n_B+1)+a}{n_B(n_A-n_B+1)}\right)\mu_{a,a}$$
$$+\frac{(n_A+1)(2-\frac{1}{2c})-n_B}{n_A-n_B+1}\sum_{a=0}^{n_B-2}\binom{n_A}{a}\binom{n_B-1}{a+1}\mu_{a,a+1}$$
$$+\sum_{a=1}^{n_B-1}\binom{n_A}{a+1}\binom{n_B}{a}\left(\frac{-\frac{\frac{1}{2}-c}{c}(n_B-a)(n_A-n_B+1)+n_A(n_B-a)-(n_B+a)}{n_B(n_A-n_B+1)}\right)\mu_{a+1,a}$$
$$+\left(\frac{c(n_A-1)-\left(\frac{1}{2}-c\right)(n_A-n_B+1)}{c(n_A-n_B+1)}\right)n_A\mu_{1,0} + \sum_{a=0}^{n_B-2}\binom{n_A}{a}\left(\frac{n_B-a-1}{n_A-n_B+1}\right)\mu_{a,n_B}$$
$$+\sum_{a=0}^{n_B-3}\sum_{b=a+2}^{n_B-1}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{(n_B-b)(2n_A-n_B+1)+(b-a-1)n_B}{n_B(n_A-n_B+1)}\right)\mu_{a,b}$$
$$+\sum_{a=n_B+2}^{n_A-1}\sum_{b=1}^{n_B-1}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{(n_B-b)(2n_A-n_B+1)+(b-a-1)n_B}{n_B(n_A-n_B+1)}\right)\mu_{a,b}$$
$$+\sum_{a=3}^{n_B}\sum_{b=1}^{a-2}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{(n_B-b)(2n_A-n_B+1)+(b-a-1)n_B}{n_B(n_A-n_B+1)}\right)\mu_{a,b}$$
$$+\sum_{a=2}^{n_A-1}\binom{n_A}{a}\frac{2n_A-n_B+1-a}{n_A-n_B+1}\mu_{a,0} + \sum_{b=0}^{n_B-2}\binom{n_B}{b}\left(\frac{n_B-2b}{n_B}\right)\mu_{n_A,b}$$
$$+\sum_{b=1}^{n_B-1}\binom{n_A}{n_B+1}\binom{n_B}{b}\left(\frac{(n_B-b)(2n_A-2n_B+1)-(n_B+b)}{n_B(n_A-n_B+1)}\right)\mu_{n_B+1,b} \tag{63}$$

It is important to determine the signs of all the terms in the above expression. It is easy to see that the first term (the largest tied profile) is positive, the next one negative. The terms with $\mu_{a,a}$, $a \in \{1,\ldots,n_B-1\}$, are negative, as well as the terms with $\mu_{a,a+1}$, $a \in$

$\{0, \ldots, n_B - 2\}$.[23] The terms with $\mu_{a+1,a}, a \in \{1, \ldots, n_B - 1\}$, are negative too.[24] The term with $\mu_{1,0}$ can be positive depending on the cost (but has the lowest possible total turnout). The remaining terms on the same line are positive. All terms on the next three lines (terms with $\mu_{a,b}$ for $a \in \{0, \ldots, n_B - 3\}, b \in \{a + 2, \ldots, n_B - 1\}$; $a \in \{n_B + 2, \ldots, n_A - 1\}, b \in \{1, \ldots, n_B - 1\}$; and $a \in \{3, \ldots, n_B\}, b \in \{1, \ldots, a-2\}$) are positive.[25] The terms with $\mu_{a,0}, a \in \{2, \ldots, n_A - 1\}$ are all positive. The last term on the same line (with $\mu_{n_A,b}$) is positive for $b \in [0, \lfloor \frac{n_B}{2} \rfloor]$ and negative for $b \in [\lfloor \frac{n_B}{2} \rfloor + 1, n_B - 2]$. The terms on the last line (terms with $\mu_{n_B+1,b}, b \in \{1, \ldots, n_B - 1\}$) are all positive, since even for $b = n_B - 1$, the numerator is positive.

We can now start optimizing by setting $\mu_{a,b} = 0$ for all negative terms with total turnout smaller than $n_B + n_B$. That is, in (63) we set

$$\mu_{a,a} = 0, a \in \{0, \ldots, n_B - 1\} \tag{64}$$

$$\mu_{a,a+1} = 0, a \in \{0, \ldots, n_B - 2\} \tag{65}$$

$$\mu_{a+1,a} = 0, a \in \{1, \ldots, n_B - 1\} \tag{66}$$

Given the slack even IC for N_A at the optimum when $n_B > \lceil \frac{1}{2} n_A \rceil$ (see Table 1), we

[23]To see this, note that for $0 < c < 0.25$ we have $2 - 1/2c$ negative, which is sufficient. When $0.25 < c < 0.5$, the difference $2 - 1/2c$ is positive, but since $n_B < n_A + 1$,

$$(n_A + 1)\left(2 - \frac{1}{2c}\right) - n_B < n_B\left(2 - \frac{1}{2c}\right) - n_B = n_B\left(1 - \frac{1}{2c}\right) < 0.$$

[24]This follows, since the numerator of the expression in the parentheses multiplied by $\mu_{a+1,a}$ can be rewritten as

$$\left(1 - \frac{1}{2c}\right)(n_B - a)(n_A - n_B + 1) - (n_A + 1)(n_B - a - 1) < 0.$$

[25]The case when $b \geq a + 2$ is obvious. The next one ($a \in \{n_B + 2, \ldots, n_A - 1\}, b \in \{1, \ldots, n_B - 1\}$) follows from observing that already at $a = n_A - 1, b = 1$, the numerator is

$$\begin{aligned}(n_B - 1)(2n_A - n_B + 1) + (1 - n_A)n_B &= n_B(n_A - n_B + 3) - n_A - (n_A + 1) \\ > n_B(n_A - n_B + 3) - 2n_B - n_A &> n_B(n_A - n_B + 3) - 4n_B + 1 \\ &= n_B(n_A - n_B - 1) + 1 > 0.\end{aligned}$$

The terms for $a \in \{3, \ldots, n_B\}, b \in \{1, \ldots, a-2\}$ are all positive, since even if we take the largest $a = n_B$, the numerator is positive: $(n_B - b)(2(n_A - n_B) + 1) - n_B > 0 \Leftrightarrow b < n_B - \frac{n_B}{2(n_A - n_B)+1}$, which always holds.

must have

$$
\begin{aligned}
\mu_{n_A,n_B} < \frac{\frac{1}{2}-c}{c}\binom{n_A-1}{n_B-1}\mu_{n_B,n_B} + \left(\frac{1}{2c}-1\right)\binom{n_A-1}{n_B}\mu_{n_B+1,n_B} \\
+\frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B-1}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a} + \sum_{a=1}^{n_B-1}\binom{n_A-1}{a-1}\binom{n_B}{a}\mu_{a,a}\right) \\
-\left[\sum_{b=1}^{n_B-1}\binom{n_B}{b}\mu_{n_A,b} + \sum_{a=3}^{n_B}\sum_{b=1}^{a-2}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b}\right. \\
+\sum_{a=n_B+2}^{n_A-1}\sum_{b=1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b} + \sum_{a=1}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b} \\
+\sum_{a=1}^{n_B-1}\binom{n_A-1}{a-1}\binom{n_B}{a+1}\mu_{a,a+1} + \sum_{b=1}^{n_B-1}\binom{n_A-1}{n_B}\binom{n_B}{b}\mu_{n_B+1,b} \\
\left.+\sum_{a=2}^{n_B}\binom{n_A-1}{a-1}\mu_{a,0} + \sum_{a=n_B+1}^{n_A}\binom{n_A-1}{a-1}\mu_{a,0}\right]
\end{aligned}
\tag{67}
$$

Using (64)-(66), we obtain

$$
\begin{aligned}
\mu_{n_A,n_B} < \frac{\frac{1}{2}-c}{c}\binom{n_A-1}{n_B-1}\mu_{n_B,n_B} + \frac{\frac{1}{2}-c}{c}\binom{n_A-1}{n_B}\mu_{n_B+1,n_B} \\
+\frac{\frac{1}{2}-c}{c}\mu_{1,0} - \left[\sum_{b=1}^{n_B-1}\binom{n_B}{b}\mu_{n_A,b} + \sum_{a=3}^{n_B}\sum_{b=1}^{a-2}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b}\right. \\
+\sum_{a=n_B+2}^{n_A-1}\sum_{b=1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b} + \sum_{a=1}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b} \\
+\binom{n_A-1}{n_B-2}\mu_{n_B-1,n_B} + \sum_{b=1}^{n_B-1}\binom{n_A-1}{n_B}\binom{n_B}{b}\mu_{n_B+1,b} \\
\left.+\sum_{a=2}^{n_B}\binom{n_A-1}{a-1}\mu_{a,0} + \sum_{a=n_B+1}^{n_A}\binom{n_A-1}{a-1}\mu_{a,0}\right]
\end{aligned}
\tag{68}
$$

Replacing the LHS of this expression with (63) and re-arranging, we obtain

$$
\begin{aligned}
&\frac{n_A(n_A-1)}{n_A-n_B+1}\mu_{1,0}+\sum_{a=0}^{n_B-2}\binom{n_A-1}{a}\frac{n_B-1}{n_A-n_B+1}\mu_{a,n_B}\\
&+\sum_{a=0}^{n_B-3}\sum_{b=a+2}^{n_B-1}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{(n_B-b)(2n_A-n_B+1)+(b-a-1)n_B}{n_B(n_A-n_B+1)}+\frac{a}{n_A}\right)\mu_{a,b}\\
&+\sum_{a=n_B+2}^{n_A-1}\sum_{b=1}^{n_B-1}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{(n_B-b)(2n_A-n_B+1)+(b-a-1)n_B}{n_B(n_A-n_B+1)}+\frac{a}{n_A}\right)\mu_{a,b}\\
&+\sum_{a=3}^{n_B}\sum_{b=1}^{a-2}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{(n_B-b)(2n_A-n_B+1)+(b-a-1)n_B}{n_B(n_A-n_B+1)}+\frac{a}{n_A}\right)\mu_{a,b}\\
&+\sum_{a=2}^{n_A-1}\binom{n_A}{a}\left(\frac{2n_A-n_B+1-a}{n_A-n_B+1}+\frac{a}{n_A}\right)\mu_{a,0}+\sum_{b=0}^{n_B-2}\binom{n_B}{b}\frac{2(n_B-b)}{n_B}\mu_{n_A,b}\\
&+\sum_{b=1}^{n_B-1}\binom{n_A}{n_B+1}\binom{n_B}{b}\left(\frac{(n_B-b)(2n_A-2n_B+1)-(n_B+b)}{n_B(n_A-n_B+1)}+\frac{n_B+1}{n_A}\right)\mu_{n_B+1,b}<\\
&\frac{\frac{1}{2}-c}{c}\binom{n_A-1}{n_B}\frac{n_B-1}{n_A-n_B+1}\mu_{n_B,n_B}+\frac{\frac{1}{2}-c}{c}\binom{n_A-1}{n_B}\mu_{n_B+1,n_B}\\
&+\frac{\frac{1}{2}-c}{c}(n_A+1)\mu_{1,0}+\sum_{a=n_B+1}^{n_A-1}\binom{n_A}{a}\left(\frac{a+1-n_B}{n_A-n_B+1}\right)\mu_{a,n_B}\\
&-\left[\binom{n_B}{n_B-1}\mu_{n_A,n_B-1}+\binom{n_A-1}{n_B-2}\mu_{n_B-1,n_B}\right] \qquad (69)
\end{aligned}
$$

Notice that all terms to the left of the inequality sign are positive and enter (63) with positive signs. The terms to the right of the inequality sign are all positive except the last parenthesis. Since those terms in the parentheses are not restricted by (63) (due to our constraint substitution), we optimally set them equal to zero. In addition we set $\mu_{1,0}=0$ since this allows to increase the remaining terms on the RHS that have larger turnout.

Taking into account the signs of the terms in (63), and given (69), we see that the RHS of (63) is optimized whenever we increase the RHS of (69). Therefore, in the optimum the sum of the terms on the LHS of (69) is as small as possible. It cannot be zero, though, due to the binding odd IC constraint for N_A (58). Indeed, this constraint determines the maximal allowed increase to μ_{n_B,n_B} via the sum of $\sum_{a=n_B+1}^{n_A-1}\mu_{a,n_B}$ and $\sum_{a=0}^{n_B-2}\mu_{a,n_B}$ (taken with appropriate coefficients).

Hence in the optimum, the support of the equilibrium distribution only includes the profiles of the form (a,n_B) for $a\in\{0,\dots,n_B-2\}\cup\{n_B,\dots,n_A\}$. Therefore, the optimal probability of the largest profile is

$$\mu_{n_A,n_B} = \frac{\frac{1}{2}-c}{c}\frac{1}{n_A-n_B+1}\binom{n_A}{n_B}\mu_{n_B,n_B} + \sum_{a=n_B+1}^{n_A-1}\binom{n_A}{a}\left(\frac{n_B-a-1}{n_A-n_B+1}\right)\mu_{a,n_B}$$
$$+\sum_{a=0}^{n_B-2}\binom{n_A}{a}\left(\frac{n_B-a-1}{n_A-n_B+1}\right)\mu_{a,n_B} \tag{70}$$

The only remaining constraint is that the total sum of probabilities is one, which, given (70), can be written as

$$\sum_{a=0}^{n_B-2}\binom{n_A}{a}\mu_{a,n_B} + \sum_{a=n_B}^{n_A-1}\binom{n_A}{a}\mu_{a,n_B} + \frac{\frac{1}{2}-c}{c}\frac{1}{n_A-n_B+1}\binom{n_A}{n_B}\mu_{n_B,n_B}$$
$$+\sum_{a=n_B+1}^{n_A-1}\binom{n_A}{a}\left(\frac{n_B-a-1}{n_A-n_B+1}\right)\mu_{a,n_B} + \sum_{a=0}^{n_B-2}\binom{n_A}{a}\left(\frac{n_B-a-1}{n_A-n_B+1}\right)\mu_{a,n_B} = 1$$

Rewriting,

$$\frac{n_A}{n_A-n_B+1}\left(\sum_{a=0}^{n_B-2}\binom{n_A-1}{a}\mu_{a,n_B} + \sum_{a=n_B+1}^{n_A-1}\binom{n_A-1}{a}\mu_{a,n_B}\right)$$
$$+\frac{\frac{1}{2c}+n_A-n_B}{n_A-n_B+1}\binom{n_A}{n_B}\mu_{n_B,n_B} = 1$$

Using the binding odd IC constraint for N_A, (58), we obtain

$$\sum_{a=n_B+1}^{n_A-1}\binom{n_A-1}{a}\mu_{a,n_B} + \sum_{a=0}^{n_B-2}\binom{n_A-1}{a}\mu_{a,n_B} - \frac{\frac{1}{2}-c}{c}\binom{n_A-1}{n_B}\mu_{n_B,n_B} = 0$$

Substituting into the previous expression, we obtain

$$\mu_{n_B,n_B} = \frac{2c}{\binom{n_A}{n_B}} \tag{71}$$

Hence

$$\sum_{a=n_B+1}^{n_A-1}\binom{n_A-1}{a}\mu_{a,n_B} + \sum_{a=0}^{n_B-2}\binom{n_A-1}{a}\mu_{a,n_B} = \frac{(1-2c)(n_A-n_B)}{n_A} \tag{72}$$

Plugging-in these expressions into the objective function and simplifying, we rewrite (47) as

$$
\begin{aligned}
f^* &= 1 + \sum_{a=0}^{n_B-2} (n_B + a - 1)\binom{n_A}{a}\mu_{a,n_B} + \sum_{a=n_B+1}^{n_A-1} (n_B + a - 1)\binom{n_A}{a}\mu_{a,n_B} \\
&\quad + (n_B + n_B - 1)\binom{n_A}{n_B}\mu_{n_B,n_B} + (n_A + n_B - 1)\mu_{n_A,n_B} \\
&= 1 + \sum_{a=0}^{n_B-2} (n_B + a - 1)\binom{n_A}{a}\mu_{a,n_B} + \sum_{a=n_B+1}^{n_A-1} (n_B + a - 1)\binom{n_A}{a}\mu_{a,n_B} \\
&\quad + (n_B + n_B - 1)2c + \frac{n_A + n_B - 1}{n_A - n_B + 1}\left[1 - 2c + \sum_{a=n_B+1}^{n_A-1}\binom{n_A}{a}(n_B - a - 1)\mu_{a,n_B}\right. \\
&\quad \left. + \sum_{a=0}^{n_B-2}\binom{n_A}{a}(n_B - a - 1)\mu_{a,n_B}\right] \\
&= 1 + (2n_B - 1)2c + \frac{(1-2c)(n_A + n_B - 1)}{n_A - n_B + 1} + \frac{2(n_B - 1)n_A}{n_A - n_B + 1}\left[\sum_{a=0}^{n_B-2}\binom{n_A - 1}{a}\mu_{a,n_B}\right. \\
&\quad \left. + \sum_{a=n_B+1}^{n_A-1}\binom{n_A - 1}{a}\mu_{a,n_B}\right] \\
&= 4cn_B + (1-2c)\left[1 + \frac{n_A + n_B - 1}{n_A - n_B + 1} + \frac{2(n_B - 1)n_A(n_A - n_B)}{(n_A - n_B + 1)n_A}\right] \\
&= 4cn_B + 2(1-2c)n_B\frac{n_A - n_B + 1}{n_A - n_B + 1} = 2n_B
\end{aligned}
$$

So, the maximal expected turnout is twice the size of the minority.

This completes the proof of case (i), with the exception of the knife-edge case of $n_B = \lceil \frac{1}{2} n_A \rceil$. We address this case after finishing the proof of case (ii).

Now suppose $n_B < \lceil \frac{1}{2} n_A \rceil$. Then $2n_B < n_A$. Due to the odd IC for N_A and even IC for N_B binding, we can express the probability of the largest profile as

$$
\begin{aligned}
\mu_{n_A,n_B} = \frac{\frac{1}{2}-c}{c} \Bigg(& -\sum_{a=0}^{n_B-2} \binom{n_A}{a}\binom{n_B-1}{a+1}\frac{n_A+1}{n_A-n_B+1}\mu_{a,a+1} \\
& + \sum_{a=1}^{n_B} \binom{n_A}{a}\binom{n_B}{a}\mu_{a,a}\left(\frac{a(n_A+1)-n_An_B}{n_B(n_A-n_B+1)}\right)\Bigg) \\
& + \sum_{a=0}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{(n_B-b)(n_A+1)+(b-a-1)n_B}{n_B(n_A-n_B+1)}\right)\mu_{a,b} \\
& + \sum_{a=n_B+2}^{n_A-1}\sum_{b=1}^{n_B}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{n_B(n_A+b-a)-b(n_A+1)}{n_B(n_A-n_B+1)}\right)\mu_{a,b} \\
& + \sum_{a=2}^{n_B}\sum_{b=1}^{a-1}\binom{n_A}{a}\binom{n_B}{b}\left(\frac{n_B(n_A+b-a)-b(n_A+1)}{n_B(n_A-n_B+1)}\right)\mu_{a,b} \\
& + \sum_{b=1}^{n_B}\binom{n_A}{n_B+1}\binom{n_B}{b}\left(\frac{(n_A-n_B)(n_B-b)-(n_B+b)}{n_B(n_A-n_B+1)}\right)\mu_{n_B+1,b} \\
& + \frac{n_A}{n_A-n_B+1}\sum_{a=1}^{n_A-1}\binom{n_A-1}{a}\mu_{a,0} - \sum_{b=1}^{n_B-1}\binom{n_B-1}{b-1}\mu_{n_A,b} \qquad (73)
\end{aligned}
$$

On the other hand, the even IC for N_A binding implies

$$
\begin{aligned}
\mu_{n_A,n_B} = \frac{\frac{1}{2}-c}{c} & \left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a} + \sum_{a=1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{a}\mu_{a,a}\right) \\
& - \Bigg[\sum_{b=1}^{n_B-1}\binom{n_B}{b}\mu_{n_A,b} + \sum_{a=3}^{n_B}\sum_{b=1}^{a-2}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b} \\
& + \sum_{a=n_B+2}^{n_A-1}\sum_{b=1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b} + \sum_{a=1}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b}\Bigg] \\
& - \sum_{a=1}^{n_B-1}\binom{n_A-1}{a-1}\binom{n_B}{a+1}\mu_{a,a+1} - \sum_{b=1}^{n_B-1}\binom{n_A-1}{n_B}\binom{n_B}{b}\mu_{n_B+1,b} \\
& - \sum_{a=2}^{n_A}\binom{n_A-1}{a-1}\mu_{a,0} \qquad (74)
\end{aligned}
$$

Comparing these two expressions and taking into account that the odd IC for N_B is slack, we see that at the optimum,

$$\mu_{a,a+1} = 0, a \in \{0, \dots, n_B - 1\} \tag{75}$$

$$\mu_{a,a} = 0, a \in \{0, \dots, n_B - 1\} \tag{76}$$

$$\mu_{n_B+1,b} = 0, b \in \{1, \dots, n_B - 1\} \tag{77}$$

$$\mu_{a,b} = 0, a \in \{n_B + 2, \dots, n_A - 1\}, b \in \{0, \dots, n_B\} \tag{78}$$

$$\mu_{a,b} = 0, a \in \{3, \dots, n_B + 1\}, b \in \{1, \dots, a - 2\} \tag{79}$$

$$\mu_{a,b} = 0, a \in \{0, \dots, n_B - 2\}, b \in \{a + 2, \dots, n_B\} \tag{80}$$

Given (75)-(80), we can rewrite (74) as

$$\begin{aligned}\mu_{n_A,n_B} = \frac{\frac{1}{2}-c}{c}\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a} + \frac{\frac{1}{2}-c}{c}\binom{n_A-1}{n_B-1}\mu_{n_B,n_B} \\ -\sum_{b=1}^{n_B-1}\binom{n_B}{b}\mu_{n_A,b} - \sum_{a=2}^{n_B+1}\binom{n_A-1}{a-1}\mu_{a,0} - \mu_{n_A,0}\end{aligned} \tag{81}$$

We also rewrite (73) as

$$\begin{aligned}\mu_{n_A,n_B} = \\ \sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\frac{n_B(n_A-1)-a(n_A+1)}{n_B(n_A-n_B+1)}\mu_{a+1,a} - \sum_{b=1}^{n_B-1}\binom{n_B-1}{b-1}\mu_{n_A,b} \\ +\frac{\left(\frac{1}{2}-c\right)n_A}{cn_B(n_A-n_B+1)}\binom{n_A-1}{n_B-1}\mu_{n_B,n_B} + \frac{n_A}{n_A-n_B+1}\sum_{a=2}^{n_B+1}\binom{n_A-1}{a}\mu_{a,0}\end{aligned} \tag{82}$$

Now (81) and (82) imply that in the optimum

$$\mu_{n_A,b} = 0, b \in \{1, \dots, n_B - 1\} \tag{83}$$

In addition, the slack odd IC for N_B, given (75)-(80), takes the form

$$\mu_{n_A,0} + \sum_{a=2}^{n_B+1}\binom{n_A}{a}\mu_{a,0} > \frac{\frac{1}{2}-c}{c}\sum_{a=0}^{n_B-1}\binom{n_A}{a+1}\binom{n_B-1}{a}\mu_{a+1,a} \tag{84}$$

Together with $2n_B \leq n_A - 1$, this implies that at the optimum $\mu_{a,0} = 0, a \in [2, n_B+1]$, and hence the support of the distribution includes only the profiles of the form $(a+1, a), a \in [0, n_B]$, (n_B, n_B) and $(n_A, 0)$. In particular, $\mu_{n_A,n_B} = 0$, since from (84) and (81), $\mu_{n_A,0}$ offsets μ_{n_B+1,n_B} and μ_{n_B,n_B} (from the maximization point of view, the profiles with higher turnout must receive larger probability weights).

Hence we can rewrite (81) as

$$\mu_{n_A,n_B} = 0 = \frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a} + \binom{n_A-1}{n_B-1}\mu_{n_B,n_B}\right) - \mu_{n_A,0} \tag{85}$$

The probability constraint now can be written as

$$\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a} + \binom{n_A}{n_B}\mu_{n_B,n_B} + \frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a} + \binom{n_A-1}{n_B-1}\mu_{n_B,n_B}\right) = 1$$

Simplifying,

$$\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\left(\frac{n_A+(a+1)(\frac{1}{2c}-1)}{n_A}\right)\mu_{a+1,a} + \binom{n_A-1}{n_B-1}\left(\frac{n_A}{n_B}+\frac{1}{2c}-1\right)\mu_{n_B,n_B} = 1 \tag{86}$$

From (82),

$$0 = \binom{n_A}{n_B}\frac{\frac{1}{2c}-1}{n_A-n_B+1}\mu_{n_B,n_B} + \sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\left(\frac{n_B(n_A-1)-a(n_A+1)}{n_B(n_A-n_B+1)}\right)\mu_{a+1,a} \tag{87}$$

Thus

$$\mu_{n_B,n_B} = -\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\left(\frac{n_B(n_A-1)-a(n_A+1)}{n_B(\frac{1}{2c}-1)\binom{n_A}{n_B}}\right)\mu_{a+1,a} \tag{88}$$

Now we can rewrite (86) as

$$\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a}\left[\frac{n_A+(a+1)(\frac{1}{2c}-1)}{n_A} - \left(\frac{n_A}{n_B}+\frac{1}{2c}-1\right)\left(\frac{n_B(n_A-1)-a(n_A+1)}{n_A(\frac{1}{2c}-1)}\right)\right] = 1 \tag{89}$$

Simplifying,

$$\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a}\left[1+\frac{(a+1)(\frac{1}{2c}-1)-n_B(n_A-1)+a(n_A+1)}{n_A}\right.$$
$$\left.-\frac{n_B(n_A-1)-a(n_A+1)}{n_B\left(\frac{1}{2c}-1\right)}\right]=1 \tag{90}$$

In addition, the binding odd IC for N_A implies

$$\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a}\left(n_B-a-\frac{n_B}{n_A}\right)=0 \tag{91}$$

The binding even IC for N_B is implies

$$\sum_{a=1}^{n_B}\binom{n_A}{a+1}\binom{n_B-1}{a-1}\mu_{a+1,a}=\frac{\frac{1}{2}-c}{c}\binom{n_A}{n_B}\mu_{n_B,n_B} \tag{92}$$

Using these expressions together with (86) and (88), we can (after some tedious algebra) express

$$\mu_{n_B,n_B}=\frac{2c}{\binom{n_A}{n_B}\left(1+\left(\frac{1}{2c}-1\right)\left(\frac{1}{n_A-1}+\frac{n_B}{n_A}\right)\right)} \tag{93}$$

Now, from (85),

$$\begin{aligned}\mu_{n_A,0}&=\frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a}+\binom{n_A-1}{n_B-1}\mu_{n_B,n_B}\right)\\
&=\frac{\frac{1}{2}-c}{c}\left[\binom{n_A}{n_B}\mu_{n_B,n_B}\left(\frac{1}{2c}-1\right)\left(\frac{1}{n_A-1}+\frac{n_B}{n_A}\right)+\binom{n_A}{n_B}\frac{n_B}{n_A}\mu_{n_B,n_B}\right]\\
&=\frac{\frac{1}{2}-c}{c}\binom{n_A}{n_B}\mu_{n_B,n_B}\left[\frac{1}{2c}\left(\frac{1}{n_A-1}+\frac{n_B}{n_A}\right)-\frac{1}{n_A-1}\right]\\
&=\frac{2c}{\frac{n_A+\frac{1}{2c}-2}{n_A-1}+\frac{n_B\left(\frac{1}{2c}-1\right)}{n_A}}\left(\frac{1}{2c}-1\right)\left[\frac{1}{n_A-1}\left(\frac{1}{2c}-1\right)+\frac{1}{2c}\frac{n_B}{n_A}\right]\end{aligned} \tag{94}$$

Plugging-in these expressions into the objective function and simplifying, we rewrite (47) as

$$
\begin{aligned}
f^* = 1 + \sum_{a=0}^{n_B} 2a\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a} + (2n_B-1)\binom{n_A}{n_B}\mu_{n_B,n_B} + (n_A-1)\mu_{n_A,0} \\
= 1 + \sum_{a=0}^{n_B} 2a\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a} \\
-(2n_B-1)\binom{n_A}{n_B}\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\left(\frac{n_B(n_A-1)-a(n_A+1)}{n_B(\frac{1}{2c}-1)\binom{n_A}{n_B}}\right)\mu_{a+1,a} \\
+(n_A-1)\frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a}\right. \\
\left.-\binom{n_A-1}{n_B-1}\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\left(\frac{n_B(n_A-1)-a(n_A+1)}{n_B(\frac{1}{2c}-1)\binom{n_A}{n_B}}\right)\mu_{a+1,a}\right) \\
= 1 + \sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a}\left[2a-(2n_B-1)\frac{n_B(n_A-1)-a(n_A+1)}{n_B(\frac{1}{2c}-1)}\right. \\
\left.+(n_A-2n_B+2n_B-1)\frac{(a+1)(\frac{1}{2c}-1)}{n_A}-(n_A-2n_B+2n_B-1)\frac{n_B(n_A-1)-a(n_A+1)}{n_A}\right] \\
= 1 + \sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a}\left[2a-(2n_B-1)\frac{n_B(n_A-1)-a(n_A+1)}{n_B(\frac{1}{2c}-1)}\right. \\
+(2n_B-1)\frac{(a+1)(\frac{1}{2c}-1)}{n_A}-(2n_B-1)\frac{n_B(n_A-1)-a(n_A+1)}{n_A} \\
\left.+(n_A-2n_B)\frac{(a+1)(\frac{1}{2c}-1)}{n_A}-(n_A-2n_B)\frac{n_B(n_A-1)-a(n_A+1)}{n_A}\right] \\
= 1 + 2n_B - 1 + \sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a}\left[2a-2n_B+1\right. \\
\left.+(n_A-2n_B)\frac{(a+1)(\frac{1}{2c}-1)-n_B(n_A-1)+a(n_A+1)}{n_A}\right] \\
= 2n_B + \frac{n_A-2n_B}{2cn_A}\left(1+n_B-\frac{n_B}{n_A}\right)\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a} \\
= 2n_B + \frac{n_A-2n_B}{2cn_A}\left(1+n_B-\frac{n_B}{n_A}\right)\left(1-\mu_{n_A,0}-\binom{n_A}{n_B}\mu_{n_B,n_B}\right) \\
= 2n_B + \frac{n_A-2n_B}{2cn_A}\left(1+n_B-\frac{n_B}{n_A}\right) \\
\times\left(\frac{\frac{n_A+\frac{1}{2c}-2}{n_A-1}+\frac{n_B\left(\frac{1}{2c}-1\right)}{n_A}-2c\left(\frac{1}{2c}-1\right)\left[\frac{1}{n_A-1}\left(\frac{1}{2c}-1\right)+\frac{1}{2c}\frac{n_B}{n_A}\right]-2c}{\frac{n_A+\frac{1}{2c}-2}{n_A-1}+\frac{n_B\left(\frac{1}{2c}-1\right)}{n_A}}\right) \\
= 2n_B + \frac{(n_A-2n_B)(1-2c)}{1+2c\left(\frac{n_A(n_A-1)}{n_A+n_B(n_A-1)}-1\right)} \\
= 2n_B + \phi(c) \\
= n_A \times \frac{2cn_B(n_A-1)+n_B(n_A-1)+n_A(1-2c)}{2c(n_A-n_B)(n_A-1)+n_B(n_A-1)+n_A(1-2c)} \\
= n_A \times \xi(c)
\end{aligned}
$$

This completes the proof of case (ii).

Finally, there remains the knife-edge case of $n_B = \lceil \frac{1}{2} n_A \rceil$. When n_A is odd, the odd IC for N_B is binding, so the proof of case (i) given above works just the same, giving the maximal expected turnout of $2n_B$. When n_A is even, the odd IC for N_B is slack, so the proof of case (ii) directly applies. The value of the objective function at the optimum has $\phi(c) = 0$. However, in contrast to case (ii), the support of the symmetric distribution is different and in fact, may include all profiles except the tied ones with turnout less than n_B, and the profiles of the form $(a, a+1), a \in [0, n_B - 1]$. There is no simple analytic expression available for the equilibrium support, so we verified our conclusions for this case using computer simulations. □

A.3 Proof of Corollary 3

Proof. Case (ii). In this case, the only profile in the support where the majority can lose is (n_B, n_B), so

$$\pi_m = 1 - \frac{1}{2}\binom{n_A}{n_B}\mu_{n_B,n_B} = 1 - \frac{c}{1 + \left(\frac{1}{2c} - 1\right)\left(\frac{1}{n_A - 1} + \frac{n_B}{n_A}\right)}$$

Case (i). Since in the equilibrium distribution support all the profiles where the majority wins are of the form (a, n_B) for $a \in [n_B + 1, n_A]$ plus the largest tied profile, it is easy to see that

$$\pi_m = \frac{1}{2}\binom{n_A}{n_B}\mu_{n_B,n_B} + \sum_{a=n_B+1}^{n_A-1}\binom{n_A}{a}\mu_{a,n_B} + \mu_{n_A,n_B} \tag{95}$$

The first term above equals c from (71), but we do not have enough constraints to identify the remaining terms in the sum individually. The rest of the proof characterizes the bounds on these terms producing the result in the statement.

As Table 1 shows, at the optimum there are three binding constraints plus the total sum of probabilities constraint. It turns out that the first binding constraint, (58), becomes equation (72), which we repeat here for convenience:

$$\sum_{a=0}^{n_B-2}\binom{n_A - 1}{a}\mu_{a,n_B} + \sum_{a=n_B+1}^{n_A-1}\binom{n_A - 1}{a}\mu_{a,n_B} = (1 - 2c)\left(1 - \frac{n_B}{n_A}\right) \tag{96}$$

The second binding constraint, (59), becomes an identity as it does not contain any profiles from the equilibrium support. The third binding constraint, (60), reduces to the total probability constraint. Namely,

$$\sum_{a=0}^{n_B-2} \binom{n_A}{a} \mu_{a,n_B} + \binom{n_A}{n_B} \mu_{n_B,n_B} + \sum_{a=n_B+1}^{n_A} \binom{n_A}{a} \mu_{a,n_B} = 1 \tag{97}$$

From (71), $\frac{1}{2}\binom{n_A}{n_B}\mu_{n_B,n_B} = c$. Hence from the total probability constraint, $1-\pi_m-c \geq 0$ and the first inequality in the statement follows.

The two binding constraints we are left with, (96) and (97), are not enough to determine π_m even knowing μ_{n_B,n_B}.[26] Nevertheless, from (96) and (97) we can express

$$\mu_{n_A,n_B} = \frac{n_B}{n_A}(1-2c) - \sum_{a=0}^{n_B-2} \frac{a}{n_A}\binom{n_A}{a}\mu_{a,n_B} - \sum_{a=n_B+1}^{n_A-1} \frac{a}{n_A}\binom{n_A}{a}\mu_{a,n_B}$$

Note that at the optimum $\mu_{n_A,n_B} > 0$. We want to show that $\pi_m > 0.5$ for all $c \in (0, 0.5)$. Suppose by way of contradiction that $\pi_m \leq 0.5$ for some cost c in this range. Then from (95)

$$\mu_{n_A,n_B} + \sum_{a=n_B+1}^{n_A-1} \binom{n_A}{a}\mu_{a,n_B} \leq \frac{1}{2}(1-2c)$$

Correspondingly, from (97)

$$\sum_{a=0}^{n_B-2} \binom{n_A}{a}\mu_{a,n_B} \geq \frac{1}{2}(1-2c)$$

This implies that the total probability mass is greater on the lower turnout profiles than on the higher turnout ones. Denote T the expected turnout at the optimal probability distribution. From Proposition 1, $T = 2n_B$. Then

$$\begin{aligned} 2n_B &= \sum_{a=0}^{n_B-2}(a+n_B)\binom{n_A}{a}\mu_{a,n_B} + 2c\cdot 2n_B + \sum_{a=n_B+1}^{n_A}(a+n_B)\binom{n_A}{a}\mu_{a,n_B} \\ &= (1-2c)\left(\frac{1}{2}+\varepsilon\right)\bar{\mu}_L + 2c\cdot 2n_B + (1-2c)\left(\frac{1}{2}-\varepsilon\right)\bar{\mu}_H, \end{aligned}$$

where $\bar{\mu}_L$ is the mean expected turnout at the lower turnout profiles, $\bar{\mu}_L \in (0, 2n_B-2)$; $\bar{\mu}_H$ is the mean expected turnout at the higher turnout profiles, $\bar{\mu}_H \in (2n_B+1, n_A+n_B)$,

[26] For the special case identified in Remark 3, there are just two profiles in the equilibrium support other than (n_B, n_B): $(0, n_B)$ and (n_A, n_B), hence we can derive their probabilities using (96) and (97). We obtain $\mu_{0,n_B} = (1-2c)\left(1-\frac{n_B}{n_A}\right)$, and therefore $\pi_m = \frac{n_B}{n_A} + c\left(1-\frac{2n_B}{n_A}\right)$.

and $\varepsilon \in [0, 0.5)$ is such that $\pi_m = \frac{1}{2} - \varepsilon$. But then

$$2n_B = (1-2c)\left[\left(\frac{1}{2}+\varepsilon\right)(\bar{\mu}_L + \bar{\mu}_H) - 2\varepsilon\bar{\mu}_H\right] + 2c \cdot 2n_B < 2n_B,$$

since due to $2n_B > n_A$ and turnout maximization, $\left(\frac{1}{2}+\varepsilon\right)(\bar{\mu}_L + \bar{\mu}_H) - 2\varepsilon\bar{\mu}_H < 2n_B$. Contradiction, so $\pi_m > \frac{1}{2}$. □

A.4 Proof of Proposition 2

Proof. The results follow by taking the limits of the expressions for the maximal expected turnout obtained in Proposition 1, divided by $n \equiv n_A + n_B$. To make sure the proof of Proposition 1 works in the first place, notice that the incentive compatibility constraints (49)-(52) are well-behaved for all n and bounded. □

A.5 Proof of Proposition 3

Proof. The minimum case is different, because the smallest (and so potentially optimal) profile $(0,0) \in V_T$ for all $n_A, n_B \geq 1$. Nevertheless, the symmetric distribution construction derived in the proof of Proposition 1 can be applied here just as well. Notice first that $\mu_{0,0} \geq 0$ at the optimum. Using the latter and the fact that all profile probabilities sum up to one, rewrite the objective in (35) as

$$\begin{aligned}1 - \mu_{0,0} + \sum_{\{s \mid \sum s_i = 2\}} \mu(s) + 2 \sum_{\{s \mid \sum s_i = 3\}} \mu(s) + \dots \\ +(n-2) \sum_{\{s \mid \sum s_i = n-1\}} \mu(s) + (n-1)\mu_{n_A, n_B}\end{aligned} \tag{98}$$

To minimize this expression, we want to increase $\mu_{0,0}$ as much as possible and set all remaining probabilities to their lowest possible level. Notice that profiles $(0,1)$ and $(1,0)$ are not directly present in (98), and profiles with total turnout of exactly two have the same (absolute) marginal effect on the objective as $\mu_{0,0}$.

The odd ICs for N_A and N_B, (49) and (51), respectively, restrict $\mu_{0,0}$ from above[27], and the exact bound depends on the ratio $\frac{\frac{1}{2}-c}{c}$. This ratio approaches zero when the cost increases towards 0.5, so for large enough cost, minimization requires placing the largest probability mass onto $(0,0)$ at the expense of other voting profiles (in particular, with total turnout of three or more), and so the minimal expected turnout approaches zero.

[27]If $n_A = n_B$, the right hand sides of the ICs for N_A in (49) and (51) must be a bit adjusted to have the indices in the first summation go up to $n_B - 1$ instead of n_B.

The opposite happens when the cost is close to 0, because then $\frac{\frac{1}{2}-c}{c} \to \infty$, and hence (49) and (51) both require their right hand sides being close to zero. Since $n_A \geq n_B$, this is achieved by setting $\mu_{a,a} = 0$ for $a \in \{0, \ldots, n_B - 1\}$. The probability of the largest tied profile, μ_{n_B,n_B}, can be positive at the optimum for c close to zero, because $\frac{\frac{1}{2}-c}{c}\binom{n_A-1}{n_B-1}\mu_{n_B,n_B}$ restricts $\mu_{2,0}$ from above in the even IC for N_A, (50), and $\frac{\frac{1}{2}-c}{c}\binom{n_A}{n_B}\mu_{n_B,n_B}$ restricts $\mu_{0,2}$ from above in the even IC for N_B, (52), whereas for $n_A = n_B$, μ_{n_B,n_B} is not at all restricted by the odd IC constraints (see ft.27), and for $n_A > n_B$, μ_{n_B,n_B} is only restricted by the odd IC for N_A. But in all cases, the value of the objective does not exceed 2: it is always optimal to shift the largest possible probability mass onto the profiles with total turnout between 0 to 2, so even when c is close to 0, the (possible) presence of non-zero terms with turnout exceeding 2 is compensated by their small equilibrium probabilities. The analytic expression for the missing cost-dependent part of the expected turnout, $\psi(c)$, can be now straightforwardly, though rather tediously (due to the equilibrium distribution support depending on the cost) derived using the same approach we applied in the proof of Proposition 1. For c close to 0, all IC constraints are binding, while for c close to 0.5, all but the even IC for N_B bind. □

A.6 Proof of Proposition 4

Proof. Any $t \in [f_*(c), f^*(c)]$ can be written as a linear combination of $f_*(c)$ and $f^*(c)$: $t = \lambda f_*(c) + (1-\lambda)f^*(c)$ for some $\lambda \in [0,1]$. Since $f_*(c), f^*(c)$ are expected turnouts in a min-turnout and max-turnout correlated equilibria, and the set of CE payoffs is convex, t is also an expected turnout in some correlated equilibrium given by probability distribution $\lambda\mu^* + (1-\lambda)\mu_*$. □

A.7 Proof of Lemma 2

Proof. By way of contradiction, suppose there is a correlated equilibrium with majority winning for sure. Then a profile $(0,0)$ is not in equilibrium support. The only way the IC constraint for voters in N_B can be satisfied with a positive voting cost is to restrict the total probability mass to only those profiles where no one from N_B ever votes.[28] This leaves admissible only profiles with voters from N_A voting. Denote $\nu(1_i, a-1, b)$ the equilibrium probability of a joint profile where $i \in N_A$ votes and there are $a-1$ other players from N_A voting and b players from N_B voting.[29] The IC constraint for voter i

[28]Strictly speaking, in this case the conditional probability that a voting player from N_B is pivotal is not well-defined, so the corresponding IC constraint is vacuously satisfied.

[29]This is a shorthand notation, which should be understood as a sum of probabilities of all joint profiles where i is voting and all remaining players behave as described.

from N_A (see (6)) takes the following form:

$$\sum_{a=2}^{n_A} \nu(1_i, a-1, 0) \leq \frac{\frac{1}{2}-c}{c} \nu(1_i, 0, 0)$$

At the same time, the IC constraint for any non-voter from N_B (see (5)) can be written as

$$\sum_{a=2}^{n_A} \sum_{i \in N_A} \nu(1_i, a-1, 0) \geq \frac{\frac{1}{2}-c}{c} \sum_{i \in N_A} \nu(1_i, 0, 0)$$

Clearly, both constraints cannot be satisfied simultaneously. Hence there is no correlated equilibrium with majority winning for sure. □

A.8 Proof of Proposition 6

Proof. Welfare maximizing correlated equilibria have Pr(*Majority wins*) as large as possible and expected turnout as small as possible, so profiles of the form $(a+1, a), a \in [0, n_B]$ must be in the support. The IC for non-voters in N_B is now binding at the optimum and implies

$$\begin{aligned} &\binom{n_A}{2}\mu_{2,0} + \binom{n_B-1}{1}\mu_{0,1} = \\ \frac{\frac{1}{2}-c}{c}&\left[\sum_{a=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{a}\mu_{a,a} + \sum_{a=0}^{n_B-1}\binom{n_A}{a+1}\binom{n_B-1}{a}\mu_{a+1,a}\right], \end{aligned} \tag{99}$$

hence the equilibrium profiles must also include either $(2, 0)$, or $(0, 1)$, or both. Note though that at $(0, 1)$ majority loses, so including this profile decreases the probability that majority wins as well as expected welfare. It is also important that in (99) the probability of tied profiles restricts the probability mass distributed among the welfare-optimal profiles $(a+1, a), a \in [0, n_B-1]$, so there can be at most one tied profile in the equilibrium: (n_B, n_B).

The odd IC constraint for N_A implies

$$\sum_{a=0}^{n_B}\binom{n_A-1}{a+1}\binom{n_B}{a}\mu_{a+1,a} + \binom{n_A-1}{2}\mu_{2,0} \geq \frac{\frac{1}{2}-c}{c}\left[\binom{n_A-1}{n_B}\mu_{n_B,n_B} + \binom{n_B}{1}\mu_{0,1}\right]$$

This constraint is always slack at the optimum since it does not restrict the total probability of the welfare-maximizing profiles on the left hand side.

The even IC constraint for N_A implies

$$\binom{n_A-1}{1}\mu_{2,0} \leq \frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a} + \binom{n_A-1}{n_B-1}\mu_{n_B,n_B}\right) \tag{100}$$

and is binding at the optimum due to (99). Finally, the even IC constraint for N_B implies

$$\sum_{a=1}^{n_B}\binom{n_A}{a+1}\binom{n_B-1}{a-1}\mu_{a+1,a} \leq \frac{\frac{1}{2}-c}{c}\left(\mu_{0,1} + \binom{n_A}{n_B}\mu_{n_B,n_B}\right) \tag{101}$$

This constraint can be satisfied if either $(0,1)$ or (n_B, n_B) are in the support[30], in which case the constraint is binding as long as some of the profiles $(a+1,a), a \in [1, n_B]$ are in the support. If neither $(0,1)$ nor (n_B, n_B) is in the support, the constraint requires all profiles $(a+1,a), a \in [1, n_B]$ not to be in the support as well.

Rewriting (99), we obtain

$$\binom{n_A}{2}\mu_{2,0} + \binom{n_B-1}{1}\mu_{0,1} = \frac{\frac{1}{2}-c}{c}\sum_{a=0}^{n_B-1}\binom{n_A}{a+1}\binom{n_B-1}{a}\mu_{a+1,a} \tag{102}$$

From (100),

$$(n_A-1)\mu_{2,0} = \frac{\frac{1}{2}-c}{c}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a} + \binom{n_A-1}{n_B-1}\mu_{n_B,n_B}\right) \tag{103}$$

so we can rewrite (102) as

$$\begin{aligned}\frac{\frac{1}{2}-c}{c}\left[\sum_{a=0}^{n_B-1}\left[\binom{n_A}{a+1}\binom{n_B-1}{a} - \frac{\binom{n_A}{2}}{n_A-1}\binom{n_A-1}{a}\binom{n_B}{a}\right]\mu_{a+1,a}\right] = \\ \frac{\frac{1}{2}-c}{c}\frac{\binom{n_A}{2}}{n_A-1}\left(\binom{n_A-1}{n_B}\mu_{n_B+1,n_B} + \binom{n_A-1}{n_B-1}\mu_{n_B,n_B}\right) + (n_B-1)\mu_{0,1}\end{aligned}$$

or

$$\begin{aligned}\frac{\frac{1}{2}-c}{2c}n_A\mu_{1,0} + \frac{\frac{1}{2}-c}{2c}\sum_{a=1}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\left(1-a\left(1+\frac{2}{n_B}\right)\right)\mu_{a+1,a} = \\ \frac{\frac{1}{2}-c}{c}\binom{n_A}{n_B}\frac{n_B}{2}\mu_{n_B,n_B} + (n_B-1)\mu_{0,1}\end{aligned} \tag{104}$$

Notice that all terms in the sum on the first line are negative if profiles $(a+1,a), a \in [1, n_B]$ are in the support.

[30]It is not optimal to include both profiles in the support, because then there is simultaneously a decrease in the probability majority wins and an increase in the expected turnout.

The total probability constraint takes the following form:

$$\sum_{a=0}^{n_B} \binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a} + \binom{n_A}{2}\mu_{2,0} + \binom{n_B}{1}\mu_{0,1} + \binom{n_A}{n_B}\mu_{n_B,n_B} = 1$$

Plugging in the expression for $\mu_{2,0}$ from (103), we obtain

$$\begin{aligned}&\sum_{a=0}^{n_B} \binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a}\left[1 + \frac{(\frac{1}{2}-c)(a+1)}{2c}\right]\\ &+\binom{n_A}{n_B}\mu_{n_B,n_B}\left[\frac{n_B(\frac{1}{2}-c)}{2c}+1\right] + n_B\mu_{0,1} = 1\end{aligned} \tag{105}$$

Using the total probability constraint, the total welfare minus the fixed n_B term can be written as follows

$$\begin{aligned}W - n_B = (n_A - n_B)\left[1 - \frac{1}{2}\binom{n_A}{n_B}\mu_{n_B,n_B} - n_B\mu_{0,1}\right]\\ -c\left[\sum_{a=0}^{n_B}(2a+1)\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a} + 2\binom{n_A}{2}\mu_{2,0} + 2n_B\binom{n_A}{n_B}\mu_{n_B,n_B} + n_B\mu_{0,1}\right]\end{aligned}$$

Suppose all profiles $(a+1,a), a \in [1, n_B]$ are in the support, then IC constraint (101) is binding:

$$\sum_{a=0}^{n_B} a\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a} = \frac{\frac{1}{2}-c}{c}\left(n_B\mu_{0,1} + n_B\binom{n_A}{n_B}\mu_{n_B,n_B}\right) \tag{106}$$

Thus we can rewrite the above expression for welfare as

$$\begin{aligned}W - n_B = (n_A - n_B)\left[1 - \frac{1}{2}\binom{n_A}{n_B}\mu_{n_B,n_B} - n_B\mu_{0,1}\right]\\ -c\left[\frac{2(\frac{1}{2}-c)}{c}\left(n_B\mu_{0,1} + n_B\binom{n_A}{n_B}\mu_{n_B,n_B}\right) + 1 + \binom{n_A}{2}\mu_{2,0} + (2n_B-1)\binom{n_A}{n_B}\mu_{n_B,n_B}\right]\end{aligned}$$

Simplifying, this equals

$$= n_A - n_B - c + \binom{n_A}{n_B}\mu_{n_B,n_B}\left[c - \frac{n_A+n_B}{2}\right] + n_B\mu_{0,1}\left[2c + n_B - n_A - 1\right] - c\binom{n_A}{2}\mu_{2,0}$$

Plugging in the expression for $\mu_{2,0}$, we obtain

$$\begin{aligned}n_A - n_B - c + \binom{n_A}{n_B}\mu_{n_B,n_B}\left[c - \frac{n_A+n_B}{2} - \frac{n_B(\frac{1}{2}-c)}{2}\right] + n_B\mu_{0,1}\left[2c + n_B - n_A - 1\right]\\ -\frac{\frac{1}{2}-c}{2}\left(\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}(a+1)\mu_{a+1,a}\right)\end{aligned}$$

After tedious algebraic manipulations with the binding IC constraints, we can express the sum of all $(a+1,a)$ profile probabilities as follows.

$$\frac{\frac{1}{2}-c}{2c}\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a}=\mu_{0,1}\left[n_B-1+\left(\frac{\frac{1}{2}-c}{c}\right)^2(1+\frac{n_B}{2})\right]$$
$$+\binom{n_A}{n_B}\mu_{n_B,n_B}\left[\left(\frac{\frac{1}{2}-c}{c}\right)^2(1+\frac{n_B}{2})+\frac{n_B(\frac{1}{2}-c)}{2c}\right]$$

Using this expression together with (106) to substitute the respective terms in the formula for welfare, we obtain

$$W=n_A-c+\binom{n_A}{n_B}\mu_{n_B,n_B}\left[c-\frac{n_A+n_B}{2}-\frac{n_B(\frac{1}{2}-c)}{2}\right]+n_B\mu_{0,1}\left[2c+n_B-n_A-1\right]$$
$$-c\left[\mu_{0,1}\left[n_B-1+\left(\frac{\frac{1}{2}-c}{c}\right)^2(1+\frac{n_B}{2})\right]+\binom{n_A}{n_B}\mu_{n_B,n_B}\left[\left(\frac{\frac{1}{2}-c}{c}\right)^2(1+\frac{n_B}{2})+\frac{n_B(\frac{1}{2}-c)}{2c}\right]\right]$$
$$-\frac{\frac{1}{2}-c}{2}\frac{\frac{1}{2}-c}{c}\left(n_B\mu_{0,1}+n_B\binom{n_A}{n_B}\mu_{n_B,n_B}\right)$$

Simplifying, we can finally write

$$W^*=n_A-c+\binom{n_A}{n_B}\mu_{n_B,n_B}\left[c-\frac{n_A+n_B+2n_B(\frac{1}{2}-c)}{2}-\frac{\left(\frac{1}{2}-c\right)^2(1+n_B)}{c}\right]+$$
$$\mu_{0,1}\left[c(1+n_B)+n_B\left[n_B-n_A-1\right]-\frac{\left(\frac{1}{2}-c\right)^2(1+n_B)}{c}\right] \tag{107}$$

From the binding IC constraints we obtain

$$\frac{\frac{1}{2}-c}{2c}\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a}=\mu_{0,1}\left[n_B-1+\left(\frac{\frac{1}{2}-c}{c}\right)^2(1+\frac{n_B}{2})\right]$$
$$+\binom{n_A}{n_B}\mu_{n_B,n_B}\left[\left(\frac{\frac{1}{2}-c}{c}\right)^2(1+\frac{n_B}{2})+\frac{\left(\frac{1}{2}-c\right)n_B}{2c}\right]$$

We can now write down the total probability constraint as follows

$$\frac{c+\frac{1}{2}}{\frac{1}{2}-c}\left[\mu_{0,1}\left[n_B-1+\left(\frac{\frac{1}{2}-c}{c}\right)^2(1+\frac{n_B}{2})\right]+\binom{n_A}{n_B}\mu_{n_B,n_B}\left[\left(\frac{\frac{1}{2}-c}{c}\right)^2(1+\frac{n_B}{2})+\frac{\left(\frac{1}{2}-c\right)n_B}{2c}\right]\right]$$
$$+\frac{\left(\frac{1}{2}-c\right)^2}{2c^2}\left(n_B\mu_{0,1}+n_B\binom{n_A}{n_B}\mu_{n_B,n_B}\right)+\binom{n_A}{n_B}\mu_{n_B,n_B}\left[\frac{n_B(\frac{1}{2}-c)}{2c}+1\right]+n_B\mu_{0,1}=1$$

Simplifying,

$$\mu_{0,1}\left[\frac{n_B-(c+\frac{1}{2})}{\frac{1}{2}-c}+\frac{(c+\frac{1}{2}(n_B+1))(\frac{1}{2}-c)}{c^2}\right]$$
$$+\binom{n_A}{n_B}\mu_{n_B,n_B}\left[\frac{(c+\frac{1}{2}(n_B+1))(\frac{1}{2}-c)}{c^2}+\frac{n_B}{2c}+1\right]=1$$

We can now estimate the effects of including either $(0,1)$ or (n_B,n_B) in the equilibrium support on welfare. First, let $\mu_{0,1}=0$, then

$$\binom{n_A}{n_B}\mu_{n_B,n_B}=\frac{1}{\frac{(c+\frac{1}{2}(n_B+1))(\frac{1}{2}-c)}{c^2}+\frac{n_B}{2c}+1} \tag{108}$$

Plugging in to (107), we obtain

$$W^*_{n_B,n_B}=n_A-c+\frac{\left[c-\frac{n_A+n_B+2n_B(\frac{1}{2}-c)}{2}-\frac{\left(\frac{1}{2}-c\right)^2(1+n_B)}{c}\right]}{\frac{(c+\frac{1}{2}(n_B+1))(\frac{1}{2}-c)}{c^2}+\frac{n_B}{2c}+1} \tag{109}$$

Second, let $\mu_{n_B,n_B}=0$, then

$$\mu_{0,1}=\frac{1}{\left[\frac{n_B-(c+\frac{1}{2})}{\frac{1}{2}-c}+\frac{(c+\frac{1}{2}(n_B+1))(\frac{1}{2}-c)}{c^2}\right]} \tag{110}$$

Plugging in to (107), we obtain

$$W^*_{0,1}=n_A-c+\frac{\left[c(1+n_B)+n_B\left[n_B-n_A-1\right]-\frac{\left(\frac{1}{2}-c\right)^2(1+n_B)}{c}\right]}{\frac{n_B-(c+\frac{1}{2})}{\frac{1}{2}-c}+\frac{(c+\frac{1}{2}(n_B+1))(\frac{1}{2}-c)}{c^2}} \tag{111}$$

Comparing $W_{0,1}$ and W_{n_B,n_B}, we see that (n_B,n_B) is in the support iff

$$\frac{\left[c-\frac{n_A+n_B+2n_B(\frac{1}{2}-c)}{2}-\frac{\left(\frac{1}{2}-c\right)^2(1+n_B)}{c}\right]}{\frac{(c+\frac{1}{2}(n_B+1))(\frac{1}{2}-c)}{c^2}+\frac{n_B}{2c}+1}>\frac{\left[c(1+n_B)+n_B\left[n_B-n_A-1\right]-\frac{\left(\frac{1}{2}-c\right)^2(1+n_B)}{c}\right]}{\frac{n_B-(c+\frac{1}{2})}{\frac{1}{2}-c}+\frac{(c+\frac{1}{2}(n_B+1))(\frac{1}{2}-c)}{c^2}}$$

which is equivalent to

$$2c\left[2c-cn_A-\frac{n_B+1}{2}\right]>\frac{(\frac{1}{2}-c)c(n_B+1)\left[cn_B\left[n_B-n_A\right]+c-\frac{n_B+1}{4}\right]}{c^2(\frac{3}{2}n_B-1)-c\frac{2n_B+1}{4}+\frac{n_B+1}{8}}$$

It is straightforward to check that the denominator on the RHS is always positive for $n_B \geq 1$ and $c \in (0, 0.5)$, so we can rewrite

$$\left(2c^2(2-n_A) - c(n_B+1)\right)\left(c^2(\frac{3}{2}n_B - 1) - c\frac{2n_B+1}{4} + \frac{n_B+1}{8}\right) > \\ (\frac{c}{2} - c^2)(n_B+1)\left[cn_B\left[n_B - n_A\right] + c - \frac{n_B+1}{4}\right]$$

This expression reduces to the following quadratic inequality:

$$c^2(2-n_A)(3n_B-2) - c\left[(n_B+1)(n_B\left(\frac{3}{2} + n_A - n_B\right) - n_A) + \frac{n_A-2}{2}\right] \\ + \frac{(n_B+1)(n_B-n_A)(\frac{1}{2} - n_B)}{2} > 0 \tag{112}$$

The discriminant of (112) is

$$D \equiv (n_B+1)^2\left(n_B\left(\frac{3}{2} + n_A - n_B\right) - n_A\right)^2 + \frac{(n_A-2)^2}{4} \\ + (n_A-2)(n_B+1)\left[\frac{n_B}{2} + (n_A-n_B)(1 + 6n_B(n_B-1))\right],$$

which is always positive for $n_A > 2 \geq n_B \geq 1$, so the cutoff cost is uniquely[31] determined by

$$c_* = \min\left\{0.5, \frac{(n_B+1)(n_B(\frac{3}{2} + n_A - n_B) - n_A) + \frac{n_A-2}{2} - \sqrt{D}}{2(2-n_A)(3n_B-2)}\right\} \tag{113}$$

Hence for $0 < c < c_*$ profile (n_B, n_B) is in the equilibrium support, and the optimal welfare is given by (109). For $c_* < c < 0.5$, profile (n_B, n_B) is not in the equilibrium support, but profile $(0, 1)$ is, and the optimal welfare is given by (111). These expressions were derived under the assumption that profiles $(a+1, a), a \in [1, n_B]$ are in equilibrium support. To conclude the proof, we need to consider the case where the cost is so high that these profiles are not in the support, and constraint (101) is slack. In this case, of course, (n_B, n_B) is not in the equilibrium support.

Suppose that profiles $(a+1, a), a \in \{1, \ldots, n_B\}$ are not in the equilibrium support. Then from (104),

$$\frac{\frac{1}{2} - c}{c}\mu_{1,0} = \frac{2(n_B-1)}{n_A}\mu_{0,1}$$

[31] The other critical cost value is always negative, so outside the range of $(0, 0.5)$.

Using the total probability constraint, we can now write

$$\binom{n_A}{1}\mu_{1,0} + \binom{n_A}{2}\mu_{2,0} + \binom{n_B}{1}\mu_{0,1} = 1$$

or

$$\mu_{0,1}\left[\frac{2(n_B-1)}{\frac{\frac{1}{2}-c}{c}} + 2n_B - 1\right] = 1$$

So we obtain

$$\begin{aligned}
\mu_{0,1} &= \frac{\frac{1}{2}-c}{n_B - \left(\frac{1}{2}+c\right)} \\
\mu_{1,0} &= \frac{2c(n_B-1)}{n_A\left[n_B - \left(\frac{1}{2}+c\right)\right]} \\
\mu_{2,0} &= \frac{\left(\frac{1}{2}-c\right)(n_B-1)}{\binom{n_A}{2}\left[n_B - \left(\frac{1}{2}+c\right)\right]}
\end{aligned}$$

and the optimal welfare is

$$W^* = \frac{n_B(n_B-c)(\frac{1}{2}-c) + 2c(n_B-1)(n_A-c) + (n_A-2c)\left(\frac{1}{2}-c\right)(n_B-1)}{n_B - \left(\frac{1}{2}+c\right)} \tag{114}$$

We can now compare W^* with $W^*_{0,1}$ to obtain the condition on the cost for which $(a+1,a), a \in [1, n_B]$ are not in the support: this is so iff

$$n_A - c + \frac{(\frac{1}{2}-c)c^2\left[c(1+n_B) + n_B\left[n_B - n_A - 1\right] - \frac{\left(\frac{1}{2}-c\right)^2(1+n_B)}{c}\right]}{c^2(n_B - (c+\frac{1}{2})) + (c + \frac{1}{2}(n_B+1))(\frac{1}{2}-c)^2} <$$
$$\frac{n_B(n_B-c)(\frac{1}{2}-c) + 2c(n_B-1)(n_A-c) + (n_A-2c)\left(\frac{1}{2}-c\right)(n_B-1)}{n_B - \left(\frac{1}{2}+c\right)}$$

which is equivalent to the following cubic inequality:

$$\begin{aligned}
c^3\left(n_A + \frac{n_B-5}{2}\right) + \frac{c^2}{2}\left((n_A-n_B)(n_B-1) + 3 - n_B\right) \\
-\frac{c}{4}\left(\frac{n_B+1}{2} + (n_A-n_B)(2n_B+1)\right) + \frac{(n_A-n_B)(n_B+1)}{8} < 0
\end{aligned}$$

□

A.9 Proof of Proposition 7

Proof. The proof closely follows the proof of Proposition 1. Assume that the optimal correlated equilibrium distribution is symmetric, as defined there. Although the voting costs are heterogenous, it is easy to see that once constraints (38) hold for the players with the lowest cost in each group, and constraints (39) hold for the players with the highest cost, the incentive compatibility constraints for all players will hold automatically.

Using symmetry, we obtain the following system of four inequalities with respect to $(n_A+1)(n_B+1)$ variables of the form $\mu_{a,b}$:

$$\sum_{a=1}^{n_A-1}\sum_{b=0}^{\min\{a-1,n_B\}}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b}+\sum_{a=0}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b}\geq$$
$$\frac{\frac{1}{2}-\underline{c}_A}{\underline{c}_A}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a,a}+\sum_{a=0}^{n_B-1}\binom{n_A-1}{a}\binom{n_B}{a+1}\mu_{a,a+1}\right)\tag{115}$$
$$\sum_{a=2}^{n_A}\sum_{b=0}^{\min\{a-2,n_B\}}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b}+\sum_{a=1}^{n_B-1}\sum_{b=a+1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b}\leq$$
$$\frac{\frac{1}{2}-\bar{c}_A}{\bar{c}_A}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a}+\sum_{a=1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{a}\mu_{a,a}\right)\tag{116}$$

and

$$\sum_{a=2}^{n_A}\sum_{b=0}^{\min\{a-2,n_B-1\}}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b}+\sum_{a=0}^{n_B-2}\sum_{b=a+1}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b}\geq$$
$$\frac{\frac{1}{2}-\underline{c}_B}{\underline{c}_B}\left(\sum_{a=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{a}\mu_{a,a}+\sum_{a=0}^{n_B-1}\binom{n_A}{a+1}\binom{n_B-1}{a}\mu_{a+1,a}\right)\tag{117}$$
$$\sum_{a=2}^{n_A}\sum_{b=1}^{\min\{a-1,n_B\}}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}+\sum_{a=0}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}\leq$$
$$\frac{\frac{1}{2}-\bar{c}_B}{\bar{c}_B}\left(\sum_{a=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{a}\mu_{a,a+1}+\sum_{a=1}^{n_B}\binom{n_A}{a}\binom{n_B-1}{a-1}\mu_{a,a}\right)\tag{118}$$

We will refer to the first and the third inequality above as the odd incentive compatibility (IC) constraints, and to the second and the fourth inequality as the even IC constraints, distinguished by the group.

Since $n_A > n_B$, at the largest turnout profile μ_{n_A,n_B} all voters from N_B (as well as voters from N_A, if $n_A > n_B+1$) are dummies. Hence the even IC constraint for N_B is always binding at the optimum.

The even IC constraint for N_A requires

$$\mu_{n_A,n_B} \leq \frac{\frac{1}{2}-\bar{c}_A}{\bar{c}_A}\left(\sum_{a=0}^{n_B-1}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a}+\sum_{a=1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{a}\mu_{a,a}\right)$$
$$-\left[\sum_{b=1}^{n_B-1}\binom{n_B}{b}\mu_{n_A,b}+\sum_{a=3}^{n_B}\sum_{b=1}^{a-2}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b}\right.$$
$$\left.+\sum_{a=n_B+2}^{n_A-1}\sum_{b=1}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b}+\sum_{a=1}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A-1}{a-1}\binom{n_B}{b}\mu_{a,b}\right]$$
$$-\sum_{a=1}^{n_B-1}\binom{n_A-1}{a-1}\binom{n_B}{a+1}\mu_{a,a+1}-\sum_{b=1}^{n_B}\binom{n_A-1}{n_B}\binom{n_B}{b}\mu_{n_B+1,b}-\sum_{a=2}^{n_B}\binom{n_A-1}{a-1}\mu_{a,0}$$
$$+\frac{1}{2\bar{c}_A}\binom{n_A-1}{n_B}\mu_{n_B+1,n_B}-\sum_{a=n_B+1}^{n_A}\binom{n_A-1}{a-1}\mu_{a,0} \tag{119}$$

The binding even IC for N_B requires

$$\mu_{n_A,n_B} = \frac{\frac{1}{2}-\bar{c}_B}{\bar{c}_B}\left(\sum_{a=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{a}\mu_{a,a+1}+\sum_{a=1}^{n_B}\binom{n_A}{a}\binom{n_B-1}{a-1}\mu_{a,a}\right)$$
$$-\left[\sum_{b=1}^{n_B-1}\binom{n_B-1}{b-1}\mu_{n_A,b}+\sum_{a=3}^{n_B}\sum_{b=1}^{a-2}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}\right.$$
$$\left.+\sum_{a=n_B+2}^{n_A-1}\sum_{b=1}^{n_B}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}+\sum_{a=1}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}\right]$$
$$-\sum_{a=1}^{n_B-1}\binom{n_A}{a+1}\binom{n_B-1}{a-1}\mu_{a+1,a}-\sum_{b=1}^{n_B}\binom{n_A}{n_B+1}\binom{n_B-1}{b-1}\mu_{n_B+1,b}$$
$$-\sum_{b=2}^{n_B}\binom{n_B-1}{b-1}\mu_{0,b} \tag{120}$$

These expressions immediately imply that the *odd* IC for N_A is always binding at the optimum. To see this, notice that in the binding constraint (120) all profiles relevant for the odd IC for N_A, i.e. those where a non-voter from N_A is a dummy, have the negative sign and so must be reduced as much as possible at the optimum. The only subset of profiles where a non-voter from N_A is a dummy that is not directly restricted by (120) has the form $\sum_{a=1}^{n_A-1}\binom{n_A-1}{a-1}\mu_{a,0}$. But these profiles are restricted by (119). If the latter is binding, the restriction is trivial. Suppose not, then if we reduced all directly restricted by (120) probabilities to their lower limit of zero and the odd IC for N_A was still not binding, then constraint (119) (slack by assumption) would imply that $\mu_{n_A,n_B} < 0$.

Therefore, the odd IC for N_A must bind at the optimum, so we can write it as

$$\sum_{a=n_B+1}^{n_A-1} \binom{n_A-1}{a} \mu_{a,n_B} - \frac{\frac{1}{2}-\underline{c}_A}{\underline{c}_A} \binom{n_A-1}{n_B} \mu_{n_B,n_B}$$
$$+ \binom{n_A-1}{n_B} \binom{n_B}{n_B-1} \mu_{n_B,n_B-1} + \sum_{a=0}^{n_B-2} \binom{n_A-1}{a} \mu_{a,n_B}$$
$$+ \sum_{a=0}^{n_B-2} \binom{n_A-1}{a+1} \binom{n_B}{a} \mu_{a+1,a} + \sum_{a=2}^{n_B+1} \sum_{b=0}^{a-2} \binom{n_A-1}{a} \binom{n_B}{b} \mu_{a,b}$$
$$+ \sum_{a=n_B+2}^{n_A-1} \sum_{b=0}^{n_B-1} \binom{n_A-1}{a} \binom{n_B}{b} \mu_{a,b} + \sum_{a=0}^{n_B-3} \sum_{b=a+2}^{n_B-1} \binom{n_A-1}{a} \binom{n_B}{b} \mu_{a,b} =$$
$$\frac{\frac{1}{2}-\underline{c}_A}{\underline{c}_A} \left(\sum_{a=0}^{n_B-1} \binom{n_A-1}{a} \binom{n_B}{a} \mu_{a,a} + \sum_{a=0}^{n_B-1} \binom{n_A-1}{a} \binom{n_B}{a+1} \mu_{a,a+1} \right) \tag{121}$$

Thus for all cost thresholds the even IC for N_B and the odd IC for N_A are binding. Proceeding exactly as in the proof of Proposition 1, we can express μ_{n_A,n_B} from (120) (the binding even IC for N_B) and then substitute the term $\binom{n_A-1}{n_B-1}\mu_{n_B-1,n_B}$ using (121) (the binding odd IC for N_A). The resulting expressions are the modified versions of (60), (61), and (62), respectively:

$$\mu_{n_A,n_B} = \frac{\frac{1}{2}-\bar{c}_B}{\bar{c}_B} \left(\sum_{a=0}^{n_B-1} \binom{n_A}{a} \binom{n_B-1}{a} \mu_{a,a+1} + \sum_{a=1}^{n_B} \binom{n_A}{a} \binom{n_B-1}{a-1} \mu_{a,a} \right)$$
$$- \left[\sum_{b=1}^{n_B-1} \binom{n_B-1}{b-1} \mu_{n_A,b} + \sum_{a=2}^{n_B} \sum_{b=1}^{a-1} \binom{n_A}{a} \binom{n_B-1}{b-1} \mu_{a,b} \right.$$
$$\left. + \sum_{a=n_B+1}^{n_A-1} \sum_{b=1}^{n_B} \binom{n_A}{a} \binom{n_B-1}{b-1} \mu_{a,b} + \sum_{a=0}^{n_B-2} \sum_{b=a+2}^{n_B} \binom{n_A}{a} \binom{n_B-1}{b-1} \mu_{a,b} \right], \tag{122}$$

$$\binom{n_A-1}{n_B-1} \cdot \mu_{n_B-1,n_B} = \frac{\underline{c}_A}{\frac{1}{2}-\underline{c}_A} \left[\sum_{a=1}^{n_B+1} \sum_{b=0}^{a-1} \binom{n_A-1}{a} \binom{n_B}{b} \mu_{a,b} \right.$$
$$\left. + \sum_{a=n_B+2}^{n_A-1} \sum_{b=0}^{n_B} \binom{n_A-1}{a} \binom{n_B}{b} \mu_{a,b} + \sum_{a=0}^{n_B-2} \sum_{b=a+2}^{n_B} \binom{n_A-1}{a} \binom{n_B}{b} \mu_{a,b} \right]$$
$$- \left(\sum_{a=0}^{n_B} \binom{n_A-1}{a} \binom{n_B}{a} \mu_{a,a} + \sum_{a=0}^{n_B-2} \binom{n_A-1}{a} \binom{n_B}{a+1} \mu_{a,a+1} \right), \tag{123}$$

and

$$
\begin{aligned}
\mu_{n_A,n_B} = {} & \frac{\frac{1}{2}-\bar{c}_B}{\bar{c}_B}\left(-\sum_{a=0}^{n_B-2}\binom{n_A}{a}\binom{n_B-1}{a+1}\frac{n_A+1}{n_A-n_B+1}\mu_{a,a+1}\right. \\
& \left. + \sum_{a=1}^{n_B}\binom{n_A}{a}\binom{n_B}{a}\frac{a(n_A+1)-n_B n_A}{n_B(n_A-n_B+1)}\mu_{a,a}\right) \\
& - \left[\sum_{b=1}^{n_B-1}\binom{n_B-1}{b-1}\mu_{n_A,b} + \sum_{a=2}^{n_B}\sum_{b=1}^{a-1}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}\right. \\
& \left. + \sum_{a=n_B+1}^{n_A-1}\sum_{b=1}^{n_B}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b} + \sum_{a=0}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A}{a}\binom{n_B-1}{b-1}\mu_{a,b}\right] \\
& + \frac{\frac{1}{2}-\bar{c}_B}{\bar{c}_B}\frac{\underline{c}_A}{\left(\frac{1}{2}-\underline{c}_A\right)}\frac{n_A}{n_A-n_B+1}\left[\sum_{a=1}^{n_B+1}\sum_{b=0}^{a-1}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b}\right. \\
& \left. + \sum_{a=n_B+2}^{n_A-1}\sum_{b=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b} + \sum_{a=0}^{n_B-2}\sum_{b=a+2}^{n_B}\binom{n_A-1}{a}\binom{n_B}{b}\mu_{a,b}\right]
\end{aligned} \tag{124}
$$

The key difference between (124) and (62) is that in (124), all terms in the last square brackets have an additional multiplier $\frac{\frac{1}{2}-\bar{c}_B}{\bar{c}_B}\frac{\underline{c}_A}{\left(\frac{1}{2}-\underline{c}_A\right)}$. The equivalence between (124) and (62) holds if and only if this multiplier equals 1, that is, if and only if $\underline{c}_A = \bar{c}_B$. If these cost thresholds are different, the equilibrium probability of the largest profile, as well as other profiles in the equilibrium support, will be different than in either of the cases considered in Proposition 1, and therefore, the corresponding maximal expected turnout will be different from f^*, the maximal expected turnout in Proposition 1. How much different depends on the relation between $\underline{c}_A$ and $\bar{c}_B$. Suppose that $\underline{c}_A < \bar{c}_B$. Then a simple contradiction argument implies

$$
\frac{\frac{1}{2}-\bar{c}_B}{\bar{c}_B}\frac{\underline{c}_A}{\left(\frac{1}{2}-\underline{c}_A\right)} < 1 \tag{125}
$$

In this case, maximization implies placing a smaller probability mass on the largest turnout profile than in Proposition 1.[32] Therefore, the expected turnout is lower than in Proposition 1 for all costs satisfying this condition. It is easy to show that if $\underline{c}_A > \bar{c}_B$, the opposite inequality holds in (125) and the equilibrium distribution places a larger mass on μ_{n_A,n_B}, so the expected turnout is higher than in Proposition 1.

We will now show that if $\underline{c}_A = \bar{c}_B$, the equilibrium distribution support does not change, and the maximal expected turnout corresponds to f^* from Proposition 1.[33]

[32]Note that this is true even if $\mu_{n_A,n_B} = 0$ at the optimum, because in this case decreasing the RHS of (124) means smaller probability of the next largest profile in the equilibrium support.

[33]This is exactly so for case (i), where the expected turnout does not depend on the cost. For case(ii),

The remaining IC constraints, the even IC for N_A and the odd IC for N_B, generally exhibit more complicated bind/slack properties. Unlike the homogenous cost case, their behavior at the optimum depends on the relations between the cost thresholds. We claim, however, that if $\underline{c}_A = \bar{c}_B = c$, the IC constraints exhibit the same behavior as in Proposition 1: for $n_B > \lceil \frac{1}{2} n_A \rceil$ the even IC for N_A is slack for all admissible values of c and the remaining cost thresholds, $\bar{c}_A$ and $\underline{c}_B$, and the odd IC for N_B is binding. For $n_B < \lceil \frac{1}{2} n_A \rceil$ the even IC for N_A is binding, and the odd IC for N_B is slack.

Let $n_B > \lceil \frac{1}{2} n_A \rceil$. Comparing (120) and (119), we see that the RHS of (120) cannot be strictly greater than the RHS of (119), since if this was the case, the even IC for N_A would not hold at all at the optimum of the constrained maximization program. If the RHS of (120) is strictly less than the RHS of (119), then the even IC for N_A is slack, as we claim. The critical case is when the two RHS are the same. Since by assumption $n_B > \lceil \frac{1}{2} n_A \rceil$, we have $2n_B \geq n_A + 1$, so $2n_B + 1 > n_A$. Then the total turnout of the first term on the last line of (119), $2n_B + 1$, exceeds the total turnout of the largest term in the remaining summation on that line, which is n_A, achieved at $\mu_{n_A,0}$. Hence at the optimum

$$\frac{1}{2\bar{c}_A}\binom{n_A - 1}{n_B}\mu_{n_B+1,n_B} - \sum_{a=n_B+1}^{n_A}\binom{n_A - 1}{a - 1}\mu_{a,0} > 0$$

Since $\underline{c}_A = \bar{c}_B = c$, a simple contradiction argument implies that

$$\frac{\frac{1}{2} - \bar{c}_A}{\bar{c}_A} \leq \frac{\frac{1}{2} - \bar{c}_B}{\bar{c}_B} \tag{126}$$

At the same time, $\frac{\frac{1}{2} - \bar{c}_A}{\bar{c}_A} < \frac{1}{2\bar{c}_A}$, and the largest turnout in the parentheses on the first line of (119), $2n_B$, is less than $2n_B + 1$, the turnout of the term with coefficient $\frac{1}{2\bar{c}_A}$ on the last line of (119). Optimization implies that the effect of this latter term must exceed the effect of the former. Therefore at the optimum the RHS of (120) is strictly less than the RHS of (119), and so the even IC for N_A is slack.

which is cost dependent, we will show that the expected turnout exhibits the same cost-dependent dynamics as in Proposition 1.

Turning to the odd IC constraints, we can rewrite (117) as

$$
\begin{aligned}
&\sum_{b=0}^{n_B-1}\binom{n_B-1}{b}\mu_{n_A,b} \\
&+\sum_{a=0}^{n_B-2}\binom{n_A}{a}\binom{n_B-1}{a+1}\mu_{a,a+1}+\sum_{a=2}^{n_B+1}\sum_{b=0}^{a-2}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b} \\
&+\sum_{a=n_B+2}^{n_A-1}\sum_{b=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b}+\sum_{a=0}^{n_B-3}\sum_{b=a+2}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{b}\mu_{a,b}\geq \\
&\frac{\frac{1}{2}-\underline{c}_B}{\underline{c}_B}\left(\sum_{a=0}^{n_B-1}\binom{n_A}{a}\binom{n_B-1}{a}\mu_{a,a}+\sum_{a=0}^{n_B-1}\binom{n_A}{a+1}\binom{n_B-1}{a}\mu_{a+1,a}\right)
\end{aligned}
\tag{127}
$$

Comparing (127) with (121), we again see that, except for the terms on the first two lines of (121) and those on the first line of (127), in every remaining profile of (121) the total turnout matches exactly the total turnout in the corresponding term of (127). Since $\underline{c}_A=\bar{c}_B$, we have

$$
\frac{\frac{1}{2}-\underline{c}_A}{\underline{c}_A}\leq\frac{\frac{1}{2}-\underline{c}_B}{\underline{c}_B}
\tag{128}
$$

Then at the optimum

$$
\begin{aligned}
&\sum_{b=0}^{n_B-1}\binom{n_B-1}{b}\mu_{n_A,b}+\frac{1}{2c}\binom{n_A-1}{n_B}\mu_{n_B,n_B}\leq \\
&\sum_{a=n_B+1}^{n_A-1}\binom{n_A-1}{a}\mu_{a,n_B}+\binom{n_A-1}{n_B}\mu_{n_B,n_B} \\
&+\binom{n_A-1}{n_B}\binom{n_B}{n_B-1}\mu_{n_B,n_B-1}+\sum_{a=0}^{n_B-2}\binom{n_A-1}{a}\mu_{a,n_B}
\end{aligned}
\tag{129}
$$

This follows, since $n_B>\lceil\frac{1}{2}n_A\rceil$, so $2n_B\geq n_A+1$ and all profiles on the LHS of (129) are matched by the corresponding profiles on the RHS. Together with (128), this implies that the odd IC for N_B is binding.

We can now proceed exactly as in the proof of Proposition 1. One can even show that the expression for the probability of the largest tie remains the same:

$$
\mu_{n_B,n_B}=\frac{2c}{\binom{n_A}{n_B}},
\tag{130}
$$

where $c=\underline{c}_A=\bar{c}_B$. Substituting the expressions for the probabilities of the largest profiles into the objective function, we obtain $h^*=2n_B$. This completes the proof of case (i).

Now suppose $n_B < \lceil \frac{1}{2} n_A \rceil$, then $2n_B < n_A$. Analogously to case (ii) of Proposition 1, we have the even IC for N_A binding and the odd IC for N_B slack at the optimum, if $\underline{c}_A = \bar{c}_B = c$. Due to the odd IC for N_A and the even IC for N_B binding[34], we can express the probability of the largest profile as

$$
\begin{aligned}
\mu_{n_A,n_B} = \frac{\frac{1}{2} - c}{c} \Bigg(& - \sum_{a=0}^{n_B-2} \binom{n_A}{a} \binom{n_B - 1}{a+1} \frac{n_A + 1}{n_A - n_B + 1} \mu_{a,a+1} \\
& + \sum_{a=1}^{n_B} \binom{n_A}{a} \binom{n_B}{a} \mu_{a,a} \left(\frac{a(n_A+1) - n_A n_B}{n_B(n_A - n_B + 1)} \right) \Bigg) \\
+ \sum_{a=0}^{n_B-2} \sum_{b=a+2}^{n_B} & \binom{n_A}{a} \binom{n_B}{b} \left(\frac{(n_B - b)(n_A+1) + (b-a-1)n_B}{n_B(n_A - n_B + 1)} \right) \mu_{a,b} \\
+ \sum_{a=n_B+2}^{n_A-1} \sum_{b=1}^{n_B} & \binom{n_A}{a} \binom{n_B}{b} \left(\frac{n_B(n_A + b - a) - b(n_A+1)}{n_B(n_A - n_B + 1)} \right) \mu_{a,b} \\
+ \sum_{a=2}^{n_B} \sum_{b=1}^{a-1} & \binom{n_A}{a} \binom{n_B}{b} \left(\frac{n_B(n_A + b - a) - b(n_A+1)}{n_B(n_A - n_B + 1)} \right) \mu_{a,b} \\
+ \sum_{b=1}^{n_B} & \binom{n_A}{n_B + 1} \binom{n_B}{b} \left(\frac{(n_A - n_B)(n_B - b) - (n_B + b)}{n_B(n_A - n_B + 1)} \right) \mu_{n_B+1,b} \\
& + \frac{n_A}{n_A - n_B + 1} \sum_{a=1}^{n_A-1} \binom{n_A - 1}{a} \mu_{a,0} - \sum_{b=1}^{n_B-1} \binom{n_B - 1}{b-1} \mu_{n_A,b}
\end{aligned} \tag{131}
$$

On the other hand, the even IC for N_A binding implies

$$
\begin{aligned}
\mu_{n_A,n_B} = \frac{\frac{1}{2} - \bar{c}_A}{\bar{c}_A} & \left(\sum_{a=0}^{n_B} \binom{n_A - 1}{a} \binom{n_B}{a} \mu_{a+1,a} + \sum_{a=1}^{n_B} \binom{n_A - 1}{a-1} \binom{n_B}{a} \mu_{a,a} \right) \\
& - \Bigg[\sum_{b=1}^{n_B-1} \binom{n_B}{b} \mu_{n_A,b} + \sum_{a=3}^{n_B} \sum_{b=1}^{a-2} \binom{n_A - 1}{a-1} \binom{n_B}{b} \mu_{a,b} \\
& + \sum_{a=n_B+2}^{n_A-1} \sum_{b=1}^{n_B} \binom{n_A - 1}{a-1} \binom{n_B}{b} \mu_{a,b} + \sum_{a=1}^{n_B-2} \sum_{b=a+2}^{n_B} \binom{n_A - 1}{a-1} \binom{n_B}{b} \mu_{a,b} \Bigg] \\
& - \sum_{a=1}^{n_B-1} \binom{n_A - 1}{a-1} \binom{n_B}{a+1} \mu_{a,a+1} - \sum_{b=1}^{n_B-1} \binom{n_A - 1}{n_B} \binom{n_B}{b} \mu_{n_B+1,b} \\
& - \sum_{a=2}^{n_A} \binom{n_A - 1}{a-1} \mu_{a,0}
\end{aligned} \tag{132}
$$

Comparing (132) with (131), taking into account that $\bar{c}_A \geq c$ and the odd IC for N_B

[34] These constraints bind for case (i) just as well.

is slack, we see that at the optimum, just like in case (ii) of Proposition 1,

$$\mu_{a,a+1} = 0, a \in \{0, \ldots, n_B - 1\} \tag{133}$$

$$\mu_{a,a} = 0, a \in \{0, \ldots, n_B - 1\} \tag{134}$$

$$\mu_{n_B+1,b} = 0, b \in \{1, \ldots, n_B - 1\} \tag{135}$$

$$\mu_{a,b} = 0, a \in \{n_B + 2, \ldots, n_A - 1\}, b \in \{0, \ldots, n_B\} \tag{136}$$

$$\mu_{a,b} = 0, a \in \{3, \ldots, n_B + 1\}, b \in \{1, \ldots, a - 2\} \tag{137}$$

$$\mu_{a,b} = 0, a \in \{0, \ldots, n_B - 2\}, b \in \{a + 2, \ldots, n_B\} \tag{138}$$

Given (133)-(138), we can rewrite (132) as

$$\begin{aligned}\mu_{n_A,n_B} = \frac{\frac{1}{2} - \bar{c}_A}{\bar{c}_A} \sum_{a=0}^{n_B} \binom{n_A - 1}{a} \binom{n_B}{a} \mu_{a+1,a} + \frac{\frac{1}{2} - \bar{c}_A}{\bar{c}_A} \binom{n_A - 1}{n_B - 1} \mu_{n_B,n_B} \\ - \sum_{b=1}^{n_B-1} \binom{n_B}{b} \mu_{n_A,b} - \sum_{a=2}^{n_B+1} \binom{n_A - 1}{a - 1} \mu_{a,0} - \mu_{n_A,0}\end{aligned} \tag{139}$$

We also rewrite (131) as

$$\begin{aligned}\mu_{n_A,n_B} = &\sum_{a=0}^{n_B} \binom{n_A}{a+1} \binom{n_B}{a} \frac{n_B(n_A - 1) - a(n_A + 1)}{n_B(n_A - n_B + 1)} \mu_{a+1,a} \\ &- \sum_{b=1}^{n_B-1} \binom{n_B - 1}{b - 1} \mu_{n_A,b} + \frac{\left(\frac{1}{2} - c\right) n_A}{c n_B(n_A - n_B + 1)} \binom{n_A - 1}{n_B - 1} \mu_{n_B,n_B} \\ &+ \frac{n_A}{n_A - n_B + 1} \sum_{a=2}^{n_B+1} \binom{n_A - 1}{a} \mu_{a,0}\end{aligned} \tag{140}$$

Now (139) and (140) imply that at the optimum

$$\mu_{n_A,b} = 0, b \in \{1, \ldots, n_B - 1\} \tag{141}$$

In addition, the slack odd IC for N_B, given (133)-(138), takes the form

$$\mu_{n_A,0} + \sum_{a=2}^{n_B+1} \binom{n_A}{a} \mu_{a,0} > \frac{\frac{1}{2} - \underline{c}_B}{\underline{c}_B} \sum_{a=0}^{n_B-1} \binom{n_A}{a+1} \binom{n_B - 1}{a} \mu_{a+1,a} \tag{142}$$

Together with $2n_B \leq n_A - 1$, this implies that at the optimum $\mu_{a,0} = 0, a \in \{2, \ldots, n_B + 1\}$, and hence the support of the distribution includes only the profiles of the form $(a + 1, a), a \in \{0, \ldots, n_B\}$, (n_B, n_B) and $(n_A, 0)$. In particular, $\mu_{n_A,n_B} = 0$, since from (142) and (139), $\mu_{n_A,0}$ offsets μ_{n_B+1,n_B} and μ_{n_B,n_B} (from the maximization point of view, the profiles with higher turnout must receive larger probability weights).

Hence we can rewrite (139) as

$$\mu_{n_A,n_B} = 0 = \frac{\frac{1}{2}-\bar{c}_A}{\bar{c}_A}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a} + \binom{n_A-1}{n_B-1}\mu_{n_B,n_B}\right) - \mu_{n_A,0} \quad (143)$$

The probability constraint can now be written as

$$\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a} + \binom{n_A}{n_B}\mu_{n_B,n_B} + \frac{\frac{1}{2}-\bar{c}_A}{\bar{c}_A}\left(\sum_{a=0}^{n_B}\binom{n_A-1}{a}\binom{n_B}{a}\mu_{a+1,a} + \binom{n_A-1}{n_B-1}\mu_{n_B,n_B}\right) = 1$$

Simplifying,

$$\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\left(\frac{n_A+(a+1)(\frac{1}{2\bar{c}_A}-1)}{n_A}\right)\mu_{a+1,a} + \binom{n_A-1}{n_B-1}\left(\frac{n_A}{n_B}+\frac{1}{2\bar{c}_A}-1\right)\mu_{n_B,n_B} = 1 \quad (144)$$

From (140),

$$0 = \binom{n_A}{n_B}\frac{\frac{1}{2c}-1}{n_A-n_B+1}\mu_{n_B,n_B} + \sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\left(\frac{n_B(n_A-1)-a(n_A+1)}{n_B(n_A-n_B+1)}\right)\mu_{a+1,a} \quad (145)$$

Thus

$$\mu_{n_B,n_B} = -\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\left(\frac{n_B(n_A-1)-a(n_A+1)}{n_B(\frac{1}{2c}-1)\binom{n_A}{n_B}}\right)\mu_{a+1,a} \quad (146)$$

Now we can rewrite (144) as

$$\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a}\left[\frac{n_A+(a+1)(\frac{1}{2\bar{c}_A}-1)}{n_A} - \left(\frac{n_A}{n_B}+\frac{1}{2\bar{c}_A}-1\right)\left(\frac{n_B(n_A-1)-a(n_A+1)}{n_A(\frac{1}{2c}-1)}\right)\right] = 1 \quad (147)$$

Simplifying,

$$\sum_{a=0}^{n_B}\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a}\left[1+\frac{(a+1)(\frac{1}{2\bar{c}_A}-1)}{n_A} - \frac{[n_B(n_A-1)-a(n_A+1)][2\bar{c}_A(n_A-n_B)+n_B]}{n_A n_B \bar{c}_A\left(\frac{1}{c}-2\right)}\right] = 1 \quad (148)$$

In addition, the binding odd IC for N_A implies

$$\sum_{a=0}^{n_B} \binom{n_A}{a+1} \binom{n_B}{a} \mu_{a+1,a} \left(n_B - a - \frac{n_B}{n_A} \right) = 0 \tag{149}$$

The binding even IC for N_B is implies

$$\sum_{a=1}^{n_B} \binom{n_A}{a+1} \binom{n_B - 1}{a-1} \mu_{a+1,a} = \frac{\frac{1}{2} - c}{c} \binom{n_A}{n_B} \mu_{n_B,n_B} \tag{150}$$

Using these expressions together with (144) and (146), we can (after some algebra) express

$$\mu_{n_B,n_B} = \frac{2}{\binom{n_A}{n_B} \left(\frac{1}{c} \left[1 + \frac{1}{2\bar{c}_A(n_A - 1)} + \frac{n_B}{n_A} \left(\frac{1}{2\bar{c}_A} - 1 \right) \right] - \frac{1}{\bar{c}_A(n_A-1)} \right)} \tag{151}$$

It is interesting to compare this expression with its analogue (93) in the proof of Proposition 1. Notice that the two coincide if $c = \bar{c}_A$. From (143),

$$\begin{aligned}
\mu_{n_A,0} &= \frac{\frac{1}{2} - \bar{c}_A}{\bar{c}_A} \left(\sum_{a=0}^{n_B} \binom{n_A - 1}{a} \binom{n_B}{a} \mu_{a+1,a} + \binom{n_A - 1}{n_B - 1} \mu_{n_B,n_B} \right) \\
&= \frac{\frac{1}{2} - \bar{c}_A}{\bar{c}_A} \left[\binom{n_A}{n_B} \mu_{n_B,n_B} \frac{(n_B(n_A - 1) + n_A)\left(\frac{1}{2c} - 1\right)}{n_A(n_A - 1)} + \binom{n_A}{n_B} \frac{n_B}{n_A} \mu_{n_B,n_B} \right] \\
&= \frac{\frac{1}{2} - \bar{c}_A}{\bar{c}_A} \frac{1}{n_A} \binom{n_A}{n_B} \mu_{n_B,n_B} \left[\frac{(n_B(n_A - 1) + n_A)\left(\frac{1}{2c} - 1\right)}{n_A - 1} + n_B \right] \\
&= \frac{\left(\frac{1}{2\bar{c}_A} - 1 \right) \left(\frac{1}{2c} (\frac{n_B(n_A - 1)}{n_A} + 1) - 1 \right)}{\left(\frac{1}{2\bar{c}_A} - 1 \right) \left(\frac{1}{2c} \left(\frac{n_B(n_A-1)}{n_A} + 1 \right) - 1 \right) + n_A \frac{1}{2c} - 1}
\end{aligned} \tag{152}$$

Plugging-in these expressions into the objective function and simplifying, we rewrite the analogue of (47) for the case of heterogeneous costs as

$$
\begin{aligned}
h^* &= 1 + 2\sum_{a=0}^{n_B} a\binom{n_A}{a+1}\binom{n_B}{a}\mu_{a+1,a} + (2n_B-1)\binom{n_A}{n_B}\mu_{n_B,n_B} + (n_A-1)\mu_{n_A,0} \\
&= 1 + 2n_B\left(\frac{1}{2c}-1\right)\binom{n_A}{n_B}\mu_{n_B,n_B} + (2n_B-1)\binom{n_A}{n_B}\mu_{n_B,n_B} \\
&+ (n_A-1)\frac{\left(\frac{1}{2\bar{c}_A}-1\right)}{n_A}\binom{n_A}{n_B}\mu_{n_B,n_B}\frac{(n_B(n_A-1)+n_A)\left(\frac{1}{2c}-1\right)+n_B(n_A-1)}{n_A-1} \\
&= 1 + \binom{n_A}{n_B}\mu_{n_B,n_B}\left[\frac{n_B}{c} - 1 + \left(\frac{1}{2\bar{c}_A}-1\right)\left(\frac{1}{2c}\left(n_B(1-\frac{1}{n_A})+1\right)-1\right)\right] \\
&= 1 + \frac{2\left[\frac{n_B}{c} - 1 + \left(\frac{1}{2\bar{c}_A}-1\right)\left(\frac{1}{2c}\left(n_B(1-\frac{1}{n_A})+1\right)-1\right)\right]}{\frac{1}{c}\left[1+\frac{1}{2\bar{c}_A(n_A-1)}+\frac{n_B}{n_A}\left(\frac{1}{2\bar{c}_A}-1\right)\right]-\frac{1}{\bar{c}_A(n_A-1)}} \\
&= n_A \times \frac{2\bar{c}_A n_B(n_A-1)+n_B(n_A-1)+n_A(1-2c)}{2\bar{c}_A[n_A-n_B](n_A-1)+n_B(n_A-1)+n_A(1-2c)} \\
&= n_A \times \xi(c,\bar{c}_A)
\end{aligned}
$$

Finally, a simple proof by contradiction shows that $2n_B < h^* < n_A$ for all costs $0 < c \leq \bar{c_A} < \frac{1}{2}$. This completes the proof of case (ii).

□

A.10 Proof of Proposition 8

Proof. Set q to be a probability distribution in $\Delta(S)$ that chooses the cost-independent strategies (i.e., constant functions from types to actions) with probability 1. This allows us to write $s_i(c_i) = s_i$ for all $c_i \in (0, 0.5)$. Require in addition that for all $i \in N$, all $a_{-i} \in V_D^i$, and all $a_{-i} \in V_P^i$

$$\sum_{\{s_{-i}(c_{-i})=a_{-i}\}} q(s_i, s_{-i}|s_i(c_i) = 0) = \mu(0, a_{-i})$$

and

$$\sum_{\{s_{-i}(c_{-i})=a_{-i}\}} q(s_i, s_{-i}|s_i(c_i) = 1) = \mu(1, a_{-i}),$$

where $\mu(a) \in \Delta(S)$ is the probability distribution over joint action profiles that delivers the solution to the max turnout problem under complete information with heterogeneous costs defined by $\mathcal{C}_{(\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B)}$ (see Proposition 7). Notice that for cost-independent strategies, the summations on the LHS of the above expressions are taken over a single strategy. Then for every player i, q selects the constant-zero strategy $s_i(c_i) = 0\ \forall c_i \in (0, 0.5)$ with probability $\sum_{a_{-i}} \mu(0, a_{-i})$, and the constant-one strategy $s_i(c_i) = 1 \; \forall c_i \in (0, 0.5)$ with the complementary probability. Suppose first that $\bar{c}_B = \underline{c}_A$. Then Proposition 7 holds at *any* cost profile $c \in \mathcal{C}_{(\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B)}$ with the *same* equilibrium distribution over actions, $\mu(a) \in \Delta(S)$, because this distribution is completely determined by the cost bounds $(\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B)$ and the sizes of the groups, n_A and n_B. Therefore, none of the incentive compatibility constraints (38)-(39) is violated at an arbitrary $c \in \mathcal{C}_{(\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B)}$. Hence this is true for all admissible c, and both constraints (43)-(44) hold as well. Therefore, we can guarantee the expected turnout at least as large as $\tilde{h}^*$. If $\bar{c}_B < \underline{c}_A$, then, as discussed in the proof of Proposition 7, the maximum expected turnout exceeds $\tilde{h}^*$, so again, for any fixed cost profile $c \in \mathcal{C}_{(\underline{c}_A, \bar{c}_A, \underline{c}_B, \bar{c}_B)}$ we can satisfy conditions (38)-(39) with the same equilibrium distribution $\mu(a) \in \Delta(S)$. $\square$